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# An Exploration Into Two Combinatorial Problems

Susan Janiszewski
*University of Vermont*

# AN EXPLORATION INTO TWO COMBINATORIAL PROBLEMS

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Thesis Examination Committee:

Jeff Dimitz, Ph.D. — Advisor

Dan S. Archdeacon, Ph.D.

Christian Skalka, Ph.D. — Chairperson

Frances E. Carr, Ph.D. — Vice President for Research and Dean of Graduate Studies

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# Abstract

Tom Johnson is an American musician and composer living in Paris who regularly composes pieces based on combinatorial designs. He recently posed the following question: is there a way to construct five Room squares of side seven such that each of the thirty-five possible fill patterns is used exactly once? In this thesis we study his question and several related questions concerning the possible fill patterns in sets of Room squares of side 7 and 11.

The second problem addressed in this thesis is a conjecture by Marco Buratti, a combinatorialist from the Universitiá di Perusia, Italy. He conjectured that given $p$ a prime and a multiset $S$ containing $p-1$ non-zero elements from $Z_p$, there exists a Hamiltonian path in $K_p$ where the multiset of edge lengths is $S$. Part of the motivation behind studying this particular problem is the possible application to constructing cyclic combinatorial designs. In this thesis we completely solve the case where $S$ contains at most two distinct values.

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# Part I

# Introduction

Tom Johnson is an American musician and composer living in Paris who regularly composes pieces based on combinatorial designs. Some of his works include *Kirkman's Ladies*[15], which is based on a solution to Kirkman's 15 schoolgirls problem, *Block Design for Piano* [14], which is based on a 4-(12, 6, 10) design, and *Septet* [16], which is based on a 2-(11,5,2) design. One of his recent ideas was to compose a piece based on Room squares of side 7. In his curiosity, he posed the following question: is there a way to construct five Room squares of side seven such that each of the thirty-five possible row fill patterns is used exactly once? In this thesis we study his question and several related questions concerning the possible fill patterns in sets of Room squares of side 7 and 11.

The second problem addressed in this thesis is a conjecture by Marco Buratti, a combinatorialist from the Universitiá di Perusia, Italy. He conjectured that given $p$ a prime and a multiset $S$ containing $p-1$ non-zero elements from $Z_p$, there exists a Hamiltonian path in $K_p$ where the multiset of edge lengths is $S$. Part of the motivation behind studying this particular problem is the possible application to constructing cyclic combinatorial designs. In this thesis we completely solve the case where $S$ contains at most two distinct values.

# Part II

# Tom Johnson's Problem

The problem proposed by Tom Johnson is as follows: is it possible to construct 5 Room squares of side 7 such that each of the 35 rows has a different set of filled cells.

## 1 Background

Tom Johnson originally stated this problem in terms of Room squares, so we initially researched properties of Room squares. It was soon determined that the pattern of filled cells in certain Room squares was equivalent to other combinatorial designs and the original problem could be stated in terms of equivalent problems with related designs. Therefore, this section will provide a basic understanding of the different combinatorial design structures used in the solution of the initial problem. The first two sections discuss the basics of Room squares and various constructions of Room squares. One observation is that the fill pattern of many Room squares can be described as a balanced incomplete block design. In particular, the fill pattern of a Room square of side seven is equivalent to a Fano plane. In the general case, the idea of Hadamard Designs was useful. Based on this, the next sections give background on BIBDs, Fano planes, and Hadamard Designs. Lastly, we present a solution to the original problem whenever a large set of the equivalent BIBD exists, so this background section ends with a brief discussion of large sets of t-designs.

## 1.1 Room Squares

A *Room square of side* $n$ is an $n \times n$ array filled with the symbols from the set $S = 1, 2, ..., n, n+1$ such that the following properties hold:

1.) Every cell in the array is either empty or contains an unordered pair of symbols from $S$.

2.) Each column and each row contain each symbol from $S$ exactly once.

3.) Each possible unordered pair appears exactly once in the array.

General background on Room squares can be found in [4], [7], and [17].

A *pattern* for an $n \times n$ Room square is the $n \times n$ matrix filled with zeros and ones with the ones corresponding to the filled cells of the Room square and the zeros corresponding to the empty cells. In some cases it is more convenient to refer to the pattern of the empty cells of the Room square instead of the filled cells. For this purpose, we define the *zero-pattern* of the Room square to be the incidence matrix corresponding to the zeros in the Room square. A single row of this incidence matrix is called a *row pattern*.

It is easy to see that in a given Room square of side $n$, the ordered pairs in each row form a one-factor (or perfect matching) of $K_{n+1}$. Since every ordered pair occurs in exactly one row, the set of rows forms a one-factorization of $K_{n+1}$. Likewise, the set of columns of a given Room square must also form a one-factorization.

Two one-factorizations are said to be *orthogonal one-factorizations* if any pair of one-factors contain at most one unordered pair of symbols in common. From these observations we get that the existence of a Room square of side $n$ is equivalent to the existence of two orthogonal one-factorizations of $K_{n+1}$ where one one-factorization corresponds to the rows of the Room square and the other one-factorization corresponds to the columns. The one-factors that have no unordered pair in common will correspond to a row and column that intersect in an empty cell, while the one-factors

that have one unordered pair in common will correspond to a row and column that intersect in a filled cell. If two factors had more than one unordered pair in common, then the row and column corresponding to those factors would have to intersect in more than one place since each unordered pair appears exactly once within the Room square. This is a contradiction, so therefore for two one-factorizations to be the rows and columns of a Room square they must be orthogonal.

## 1.2 Constructions of Room Squares

The most commonly used method for constructing Room squares is the starter-adder method, also known as the method of orthogonal starters. In order to effectively use this method, it is necessary to first understand what a starter is and what makes two starters orthogonal. Information on starters can be found in [5].

Let $G$ be an abelian group of odd order $g$ with $G$ written additively. A *starter* is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ in $G$ that satisfy the following properties:

1.) $\{s_i : 1 \leq i \leq (g-1)/2\} \cup \{t_i : 1 \leq i \leq (g-1)/2\} = G \backslash \{0\}$

2.) $\{\pm(s_i - t_i) : 1 \leq i \leq (g-1)/2\} = G \backslash \{0\}$

In other words, each non-zero element of $G$ appears in exactly one unordered pair and each pair has a unique difference. If a starter $S$ has the additional properties that $s_i + t_i = s_j + t_j$ implies $i = j$ and $s_i + t_i \neq 0$ for any $i$, it is called a *strong starter*.

Starters exist in every abelian group $G$ of odd order. For example, the set defined by $P = \{\{x, -x\} : x \in G\}$ is always a starter. This special set is known as the *patterned starter*. Note that $P$ is not a strong starter since for every unordered pair $s_i + t_i = 0$.

Given two starters $S$ and $T$ defined as follows:
$S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$
$T = \{\{u_i, v_i\} : 1 \leq i \leq (g-1)/2\}$

We note that without loss of generality, it can be assumed that $s_i - t_i = u_i - v_i$ for all $i$. $S$ and $T$ are said to be *orthogonal starters* if $u_i - s_i = u_j - s_j$ implies that $i = j$ and $u_i \neq s_i$ for all $i$.

An example of two orthogonal starters are $S = \{\{2,3\},\{4,6\},\{1,5\}$ and $P = \{\{3,4\},\{6,1\},\{5,2\}\}$. Note that $P$ is the patterned starter and $S$ is a strong starter.

Given a starter $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$, an *adder* for $S$ is defined as an ordered set $A_S = \{a_1, a_2, \ldots, a_{(g-1)/2}\}$ where each of the elements $a_i$ are distinct elements of $G$ and the set $T = \{\{s_i + a_i, t_i + a_i\} : 1 \leq i \leq (g-1)/2\}$ is also a starter in the group $G$. It can be easily shown that the original starter, $S$, and the starter created by the adder, $T$, are orthogonal. Similarly, given two orthogonal starters $S$ and $T$, the adder $A_S$ (and similarly $A_T$) can be found since the relationship $a_i = t_i - s_i$ holds for all $i$.

It can be shown that if there exists a strong starter $S$ in an abelian group of odd order, then $S$ and $-S$ are orthogonal starters. Furthermore, both $S$ and $-S$ are orthogonal to $P$, the patterned starter. It follows that given a strong starter $S$ in a group $G$, a Room square exists.

A Room square can be created from two orthogonal starters using the following construction method. Let $S$ and $T$ be the orthogonal starters in $G$ as defined above and let $A_S$ be the associated adder. Let our empty $n \times n$ grid (which will be filled to form a Room square) be called $R$ and let the rows and columns of $R$ be indexed by the elements of $G$. Let $R(x, y)$ denote the cell in $R$ indexed by $x$ and $y$. Place the pair $\{x, \infty\}$ in the cell $R(x, x)$. Place the elements of $S$ in the first row so that $\{s_i, t_i\}$ appears in the cell $R(0, -a_i)$. The rest of the Room square can be filled in by placing $\{s_i + x, t_i + x\}$ in the cell $R(x, -a_i + x)$. Note that when $G = Z_n$, the Room square is filled down the forward diagonals.

Below is an example of a Room square of side 7 constructed using the method

of orthogonal starters, along with the two starters from $Z_7$ used in the construction. Note that the rows arise from translates of $S$ and the columns come from translates of $P$.

$$S = \{\{2,3\},\{4,6\},\{1,5\}\} \quad P = \{\{3,4\},\{6,1\},\{5,2\}\}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\infty 0$ | | | 15 | | 46 | 23 |
| 34 | $\infty 1$ | | | 26 | | 50 |
| 61 | 45 | $\infty 2$ | | | 30 | |
| | 02 | 56 | $\infty 3$ | | | 41 |
| 52 | | 13 | 60 | $\infty 4$ | | |
| | 63 | | 24 | 01 | $\infty 5$ | |
| | | 04 | | 35 | 12 | $\infty 6$ |

## 1.3 Balanced Incomplete Block Designs

A design is a pair $(X, A)$ where $X$ is a set of *points* and $A$ is a collection of non-empty subsets of $X$ called *blocks*. A special class of designs are called *balanced incomplete block designs*, abbreviated BIBD. A BIBD is characterized by the parameters $v, k$, and $\lambda$, where $v$ is the number of points, $k$ is the number of points in each block, and every distinct pair of points is contained in exactly $\lambda$ blocks. A good general reference for results on BIBDs is [13] and [12]. There are also two other parameters associated with BIBDs, $b$ and $r$ where $b$ is the total number of blocks in the design and $r$ is the number of blocks that a given point occurs in. The parameter $r$ is also known as the *replication number*. We denote a BIBD with a given set of parameters as a $(v, k, \lambda)$-BIBD. Since $b$ and $r$ can be calculated directly from $v, k$, and $\lambda$, it is not necessary to include them in the characterization. The formulas used to find $b$ and $r$ are as follows:

$$r = \frac{\lambda(v-1)}{k-1}$$

$$b = \frac{vr}{k} = \frac{\lambda(v^2 - v)}{k^2 - k}$$

From the above equations one can deduce that in any BIBD, $bk = vr$.

The *incidence matrix* for a given BIBD is a $b \times v$ matrix with the rows indexed by the blocks of the design and the columns are indexed by the points. Each row is filled with zeros and ones. In a given row, the ones are placed in the columns corresponding to the points in that block. The zeros are placed in the remaining cells. The incidence matrix of any BIBD has the following properties: every column contains exactly $k$ ones, every row contains exactly $r$ ones, and two distinct rows of the matrix both contain ones in exactly $\lambda$ columns. Another characteristic of incidence matrices is for any incidence matrix $A$, $AA^T = \lambda J_v + (r - \lambda)I_v$, i.e. $AA^T$ is a $v \times v$ matrix with $r$ down the diagonal and $\lambda$ off the diagonal.

BIBDs with the additional property that $b = v$ are termed *symmetric designs*. Equivalently, we could say that a symmetric BIBD has the property $k = r$. It is fairly easy to show that in a symmetric BIBD not only does every pair of points occur in $\lambda$ blocks together, but every two blocks intersect in exactly $\lambda$ points. Another property of symmetric designs that will be used in this paper is that the complement of a symmetric BIBD is also a symmetric BIBD.

A special class of symmetric BIBDs are projective planes. A *projective plane of order* $n$ is an $(n^2 + n + 1, n + 1, 1)$-BIBD. Projective planes of order $n$ exist for every prime power $n$. In this paper, the only projective plane of consequence is the (7,3,1)-BIBD, also known as the Fano plane.

## 1.4 Hadamard Matrices and Designs

A *Hadamard matrix* of order $n$ is an $n \times n$ matrix $H$ in which every entry is $\pm 1$ such that $HH^T = nI_n$. Good references for information on Hadamard matrices are

[3] and [13]. Below is an example of a Hadamard matrix of order 4:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

A Hadamard matrix is *standardized* if the first row and the first column are all ones. Note that the above Hadamard matrix is standardized. It can be easily observed that any Hadamard matrix can be standardized since the result of multiplying any row or column of a Hadamard matrix by $-1$ is another Hadamard matrix.

Let $r_i$ denote a row of a Hadamard matrix. In the matrix $HH^T$, the entry $(i, j)$ corresponds to the inner product of rows $r_i$ and $r_j$. By definition, the entries off of the diagonal of $HH^T$ are all zero. It therefore follows that the inner product of any two rows $r_i$ and $r_j$ where $i \neq j$ must be zero.

It is easy to prove that if a Hadamard matrix of order $n > 2$ exists, then $n \equiv 0 \pmod 4$ (see [13]). A famous open conjecture of Jacques Hadamard's is that there exists a Hadamard matrix for every order $n \equiv 0 \pmod 4$.

There is also an equivalence between Hadamard matrices and certain symmetric BIBDs. There exists a Hadamard matrix if and only if there exists a symmetric $(4m-1, 2m-1, m-1)$-BIBD. To demonstrate this, let $H$ be a standardized Hadamard matrix of order $4m$. We form what is known as the core of a standardized Hadamard matrix by deleting the first row and the first column of $H$. The core can further be transformed into the incidence matrix $M$ of a symmetric $(4m-1, 2m-1, m-1)$-BIBD by replacing every entry of $-1$ with a 0. This type of BIBD is known as a *Hadamard design*. For example, below is a Hadamard matrix of order 8, $H$, as well as the incidence matrix, $M$, and the block representation, $D$, of the corresponding

$(7, 3, 1)$-BIBD.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$D = \{\{0,1,2\},\{0,3,4\},\{0,5,6\},\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}$$

## 1.5 Large Sets of Designs

We begin with a generalization of a BIBD: a *t-(v,k,λ)* design is a pair $(X, B)$ where $X$ and $B$ have the following properties:

1. $X$ is a set of $v$ elements, called *points*.
2. $B$ is a collection of $k$-subsets of $X$, called *blocks*.
3. Every $t$-subset can be found in exactly $\lambda$ blocks.

A $t$-design is *simple* if it contains no repeated blocks. Previously we discussed $(v, k, \lambda)$-BIBDs, which are also $2-(v, k, \lambda)$ designs. In particular, each pair of points (which gives $t = 2$) appears in exactly $\lambda$ blocks. We also note that given a *t-(v,k,λ)*-design, a *t-(v,v-k,λ′)*-design can be created by taking the complement of each block.

In this new design, $\lambda'$ can be calculated using the following relationship:

$$\lambda' = \frac{\lambda\binom{v-k}{k}}{\binom{k}{t}}$$

A *large set* of $t-(v,k,\lambda)$ designs is a partition of all the $\binom{v}{k}$ subsets of size $k$ chosen from $v$ points into $t-(v,k,\lambda)$-designs. Notation for a large set of $t-(v,k,\lambda)$-designs is LS $t-(v,k,\lambda)$. As with individual $t$-designs, taking the complements of the blocks in a large set of $t-(v,k,\lambda)$ designs will yield a large set of $t-(v,v-k,\lambda')$ designs. Since the large set is a partition of the possible blocks, it follows that each of the designs in the large set must be a simple design and each block appears in only one design. Therefore the number of total designs $N$ in the large set must be the number of possible blocks divided by the number of blocks per design. When $N$ is not known, it can be calculated using the following equation:

$$N = \binom{v-t}{k-t/\lambda}$$

See [9] for more information on large sets of $t$-designs.

# 2 Solution for Room Squares of Side 7

## 2.1 Possible Room Squares of Side 7

When presented with the problem of determining whether or not a set of five Room squares of side 7 containing all distinct row patterns existed, we first looked at characterizing all possible Room squares of side 7. Below is a list of the 6 different one-factorizations of $K_8$. A proof that this is the complete set of one-factors of $K_8$ can be found in [17]. These factors were then used to create all possible Room squares of side 7.

| $F_1$ | $F_2$ | $F_3$ |
|---|---|---|
| $A = 01\ 23\ 45\ 67$ | $A = 01\ 23\ 45\ 67$ | $A = 01\ 23\ 45\ 67$ |
| $B = 02\ 13\ 46\ 57$ | $B = 02\ 13\ 46\ 57$ | $B = 02\ 13\ 46\ 57$ |
| $C = 03\ 12\ 47\ 56$ | $C = 03\ 12\ 47\ 56$ | $C = 03\ 12\ 47\ 56$ |
| $D = 04\ 15\ 26\ 37$ | $D = 04\ 15\ 26\ 37$ | $D = 04\ 16\ 25\ 37$ |
| $E = 05\ 14\ 27\ 36$ | $E = 05\ 14\ 27\ 36$ | $E = 05\ 17\ 26\ 34$ |
| $F = 06\ 17\ 24\ 35$ | $F = 06\ 17\ 25\ 34$ | $F = 06\ 14\ 27\ 35$ |
| $G = 07\ 16\ 25\ 34$ | $G = 07\ 16\ 24\ 35$ | $G = 07\ 15\ 24\ 36$ |

| $F_4$ | $F_5$ | $F_6$ |
|---|---|---|
| $A = 01\ 23\ 45\ 67$ | $A = 01\ 23\ 45\ 67$ | $A = 01\ 23\ 45\ 67$ |
| $B = 02\ 13\ 46\ 57$ | $B = 02\ 13\ 46\ 57$ | $B = 02\ 14\ 36\ 57$ |
| $C = 03\ 12\ 47\ 56$ | $C = 03\ 14\ 27\ 56$ | $C = 03\ 16\ 25\ 47$ |
| $D = 04\ 16\ 27\ 35$ | $D = 04\ 16\ 25\ 37$ | $D = 04\ 17\ 26\ 35$ |
| $E = 05\ 17\ 26\ 34$ | $E = 05\ 17\ 26\ 34$ | $E = 05\ 12\ 37\ 46$ |
| $F = 06\ 12\ 35\ 47$ | $F = 06\ 12\ 35\ 47$ | $F = 06\ 15\ 27\ 34$ |
| $G = 07\ 15\ 24\ 36$ | $G = 07\ 15\ 24\ 36$ | $G = 07\ 13\ 24\ 56$ |

Two room squares are said to be *isomorphic* if one can be changed into the other through a series of row and column permutation. It was shown in [17] that there are only 10 non-isomorphic Room squares of side 7. These were denoted as $R_{1,1}, R_{1,4}, R_{1,5}, R_{1,6}, R_{4,1}, R_{4,4}, R_{4,5}, R_{5,1}, R_{5,4}, R_{6,1}$ where $R_{i,j}$ is a square with $F_i$ as its row factorization and $F_j$ as its column factorization. Two Room squares are equivalent if one Room square is isomorphic to a transpose of the other. So from the above we see that there are a total of six non-equivalent Room squares of side seven. We give these below.

$R_{1,1}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | | 45 | 67 | | | 23 |
| 57 | 02 | | | | 13 | 46 |
| | 56 | 03 | 12 | | 47 | |
| | 37 | | 04 | 26 | | 15 |
| 36 | 14 | 27 | | 05 | | |
| 24 | | | 35 | 17 | 06 | |
| | | 16 | | 34 | 25 | 07 |

$R_{1,6}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | 67 | | | | 45 | 23 |
| 57 | 02 | 46 | | 13 | | |
| | | 03 | | 47 | 12 | 56 |
| | 15 | | 04 | 26 | 37 | |
| 36 | | | 27 | 05 | | 14 |
| 24 | | 17 | 35 | | 06 | |
| | 34 | 25 | 16 | | | 07 |

$R_{1,4}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | | 45 | 67 | | 23 | |
| 57 | 02 | | 13 | | | 46 |
| | 56 | 03 | | | 47 | 12 |
| | 37 | | 04 | 26 | 15 | |
| 36 | 14 | 27 | | 05 | | |
| 24 | | | | 17 | 06 | 35 |
| | | 16 | 25 | 34 | | 07 |

$R_{4,4}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | 67 | | | 23 | | 45 |
| | 02 | 46 | | | 57 | 13 |
| 56 | | 03 | | 47 | 12 | |
| | 35 | 27 | 04 | 16 | | |
| | | | 17 | 05 | 34 | 26 |
| 37 | 14 | | 25 | | 06 | |
| 24 | | 15 | 36 | | | 07 |

$R_{1,5}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | | | | 67 | 23 | 45 |
| 57 | 02 | 46 | | | | 13 |
| | 56 | 03 | | 12 | 47 | |
| | 37 | | 04 | | 15 | 26 |
| 36 | 14 | | 27 | 05 | | |
| 24 | | 17 | 35 | | 06 | |
| | | 25 | 16 | 34 | | 07 |

$R_{4,5}$

| | | | | | | |
|---|---|---|---|---|---|---|
| 01 | 67 | | | 23 | 45 | |
| | 02 | 57 | | | 13 | 46 |
| 56 | | 03 | | 47 | | 12 |
| | | | 04 | 16 | 27 | 35 |
| | 34 | 26 | 17 | 05 | | |
| 37 | | 14 | 25 | | 06 | |
| 24 | 15 | | 36 | | | 07 |

Studying the patterns in the above Room squares, it was noted that each pair of rows had inner product of two. This led us to the following theorem, which characterizes all Room squares of side 7.

**Theorem 1** *A 7×7 Room square exists with pattern $P$ $\iff$ $P$ is the incidence matrix of a $(7, 4, 2)$ design.*

**Proof:** Assume that we have a 7×7 Room square with pattern $P$. In order to show it is the incidence matrix of a $(7, 4, 2)$ design it is necessary to show that the pattern has constant column sum of four, constant row sum of four, and the property that the inner product of any two rows is two. By the definition of a Room square, each row sum and each column sum of $P$ must be four. As stated above, there are only six non-equivalent Room squares of side seven. Each of these was manually checked that each pair of rows indeed had inner product of two. Any Room square isomorphic to one of the standard Room squares must be obtainable by some combination of row

permutations and column permutations on the standard square. Obviously, any set of row permutations will preserve the property that the inner product between two rows is two. Likewise, column permutations do not change the value of the inner product between two rows. Therefore, every Room square of side seven must have the property that its fill pattern has inner product of two since every Room square is equivalent to one of the above six squares.

To prove the converse, assume that we have a fill pattern $P$ that is a $(7, 4, 2)$ design. The (7,4,2)-BIBD is the complement of the (7,3,1)-BIBD, also known as the Fano plane. It is known that there is only one Fano plane, up to isomorphism. Therefore, there is only one (7,4,2)-BIBD up to isomorphism as well. It follows that given $P$, a (7,4,2)-BIBD, a permutation $\sigma$ can be found that takes it to the (7,4,2)-BIBD that corresponds to the fill pattern of $R_{1,1}$. The inverse permutation can then take the filled Room square back to the Room square with a fill pattern corresponding to $P$. ■

**Theorem 2** *Every pattern $P$ with the above properties can have six non-equivalent Room squares of side seven placed on it.*

**Proof:**

By the observations in the above proof, a fill pattern $P$ with the above properties is equivalent to the complement of the Fano plane and, up to isomorphism, there is only one (7,4,2)-BIBD. Let $D_1, D_2, \cdots, D_6$ be the (7,4,2)-BIBDs corresponding to the fill patterns for the six standard non-equivalent Room squares and let $D$ be the (7,4,2)-BIBD corresponding to the fill pattern $P$. Since all (7,4,2)-BIBDs are isomorphic, there exists a permutation that takes $D$ to $D_i$ for any $i$. Therefore, the inverse permutation can take the standard Room square back to the square with the original fill pattern. Each of the six standard squares can be placed into the square

with pattern $P$, giving six non-isomorphic Room squares with that fill pattern. ■

Given the above theorems, it is now clear that Tom Johnson's original question about fill patterns in Room squares of side 7 is really just a question about the structure of sets of $(7, 4, 2)$-BIBDs, (or equivalently, $(7, 3, 1)$-BIBDs). In the remainder of this chapter we concern ourselves with this question. Specifically, we will be interested in the number of different blocks that can occur in sets of $i$ distinct Fano planes for $i = 2, 3, 4, 5$, and 6.

## 2.2 Classes of Fano Planes

It follows from Theorem 1 that every zero-pattern of a Room square of side seven is the incidence matrix of a $(7, 3, 1)$-BIBD (also denoted a $2 - (7, 3, 1)$-design). It follows that a solution to Tom Johnson's initial problem would be equivalent to the existence of a large set of $(7, 3, 1)$-BIBDs. It was proved in 1850 by Cayley that such a set does not exist [1]. Furthermore, it is easy to show that the largest set of Fano planes that contain all distinct blocks is 2. Since the non-existence of a large set of Fano planes solves the initial problem, the next questions to be asked are how many distinct blocks can be contained in a set of Fano planes and how many Fano planes are required to achieve all of the distinct blocks. To answer these questions, I started investigating properties of the Fano plane. Many of my searches involving the computer software package *Mathematica*.

**Lemma 3** *There are thirty different Fano planes.*

**Proof:** This observation was first made using an exhaustive computer search. I first determined how many ways we could choose the four rows of a Room square that had the first column filled and still preserve the property of every two rows having inner

product of two. I then determined how many ways the three rows of the Room square that contained an empty cell in the first column could be chosen. Lastly, I took every combination of the four rows and three rows and determined which combinations yielded a Room square. There were exactly thirty such combinations, giving thirty possible Room squares of side 7. This is equivalent to thirty different Fano planes by Theorem 1.

This observation can also be proved using a much shorter and more elegant combinatorial approach. Assume the symbols $\{a, b, c, d, e, f, g\}$ are the points in the Fano plane. The points $a$ and $b$ must appear in a block together, and there are 5 ways to choose the last point in that block. Choose one of the points not yet in a block with $a$. These two points must be a block together, and there are 3 ways to choose the last element of that block. The third block containing $a$ is now also uniquely determined. At this point every block with the point $a$ is determined. The point $b$ is already in one block, so choose one of the elements, $x$, that it is not yet in a block with $b$. There are three points left that $b$ has not been in a block with, but one of these has already appeared in a block with $x$, leaving only two choices to finish this particular block. At this point there is only one way to fill in the remaining blocks, so the plane is uniquely determined. Therefore, there are $5 * 3 * 2 = 30$ different ways of constructing the Fano plane. ■

**Lemma 4** *If $F_1, F_2$, and $F_3$ are three Fano planes with $|F_1 \cap F_2| = 1$ and $|F_1 \cap F_3| = 1$, then $|F_2 \cap F_3| = 1$.*

**Proof:** There are two cases to check. The first is when $F_1 \cap F_2 = F_1 \cap F_3$ and the second is when $F_1 \cap F_2 \neq F_1 \cap F_3$.

First we examine case 1. Without loss of generality we can define $F_1$ as below and assume that $F_2$ and $F_3$ intersect $F_1$ in block $\{a, b, c\}$.

$$F_1 = \{\{a,b,c\},\{a,d,e\},\{a,f,g\},\{b,d,f\},\{b,e,g\},\{c,e,f\},\{c,d,g\}\}$$

There are only two ways to complete a Fano plane containing the block $\{a,b,c\}$ that only intersects $F_1$ in that block. We label these $F_2$ and $F_3$.

$$F_1 = \{\{a,b,c\},\{a,d,e\},\{a,f,g\},\{b,d,f\},\{b,e,g\},\{c,e,f\},\{c,d,g\}\}$$
$$F_2 = \{\{a,b,c\},\{a,d,f\},\{a,e,g\},\{b,d,g\},\{b,e,f\},\{c,d,e\},\{c,f,g\}\}$$
$$F_3 = \{\{a,b,c\},\{a,d,g\},\{a,e,f\},\{b,d,e\},\{b,f,g\},\{c,d,f\},\{c,e,g\}\}$$

It is easy to check that the only block that $F_2$ and $F_3$ have in common is $\{a,b,c\}$. It follows that given any two planes $F_2$ and $F_3$, if $F_1 \cap F_2 = a$ and $F_1 \cap F_3 = a$ where $a$ denotes a block, then $F_2 \cap F_3 = a$, i.e. any planes that intersect with $F_1$ in exactly one block and intersect with each other in the same block in which they intersect $F_1$ intersect only in that block.

Now, given two planes that intersect with $F_1$ in different blocks, it is necessary to prove that their intersection is also of size one. We can assume without loss of generality the two blocks are $\{a,b,c\}$ and $\{a,d,e\}$ since any two blocks of the Fano plane intersect in a unique point. The two other planes that contain $\{a,b,c\}$ and intersect once with $F_1$ from above are $F_2$ and $F_3$. The two planes that contain $\{a,d,e\}$ and intersect $F_1$ in exactly one block are $F_4$ and $F_5$ as below.

$$F_4 = \{\{a,b,f\},\{a,c,g\},\{a,d,e\},\{b,c,e\},\{b,d,g\},\{c,d,f\},\{e,f,g\}\}$$
$$F_5 = \{\{a,b,g\},\{a,c,f\},\{a,d,e\},\{b,c,d\},\{b,e,f\},\{c,e,g\},\{d,f,g\}\}$$

Checking the four possible combinations we see that $F_2$ and $F_4$ intersect only in $\{b,d,g\}$, $F_2$ and $F_5$ intersect only in $\{b,e,f\}$, $F_3$ and $F_4$ intersect only in $\{c,d,f\}$, and $F_3$ and $F_5$ intersect only in $\{c,e,g\}$. Since we started with arbitrary blocks, it follows that any two planes that intersect with $F_1$ in exactly one block intersect with each other in exactly one block. ■

From Lemma 4 we now can define two planes to be in the same *class* if they share exactly one block together. With this definition, we will show that the Fano planes can be divided into two distinct classes which each class having certain properties. These properties include: each class is well-defined, each class contains 15 of the 30 total planes; two planes in different classes are either disjoint or intersect in exactly three blocks; each plane is disjoint from exactly eight planes in the other class and has intersection of 3 with seven planes in the other class; and these seven planes can be identified with the points in a Fano plane in a natural way. We will refer to the two classes of Fano planes Class $A$ and Class $B$.

The classes were characterized and these properties first observed by utilizing computer search methods and the software program *Mathematica* (see Appendix A for code). The $\binom{7}{3}$ total blocks were numbered from one to thirty-five, and each Fano plane was written as the set of seven numbers that corresponded to the blocks in that particular plane. I then created a graph where the points were the Fano planes, and edges were drawn between planes that had intersection of exactly one block. The program that searches for the maximum clique returned a clique of size fifteen. It was then checked that the remaining fifteen planes also formed a clique in this graph. Lastly, the degree of each vertex was found to be fourteen, which shows that if we take each set of fifteen to be our classes, there are no edges between classes. Thus we have that there are two sets of fifteen planes where within each set each pair of planes intersects in exactly one block, giving us well-defined classes.

Four blocks uniquely determine a Fano plane, so any two planes that intersect in four or more blocks must in fact be the same plane. Any two planes that intersect in exactly one plane are defined to be in the same class, leaving two planes in separate classes to intersect in zero, two, or three blocks. Using the seven number representations of the Fano planes, a graph was defined with the points as the planes

and edges being drawn between two planes that were disjoint from each other. The degree of each vertex was eight, therefore each plane is disjoint from exactly eight planes in the other class. A second graph was defined, where this time edges were drawn between planes that had exactly three blocks in their intersection. This graph was found to be regular of degree seven, thus showing that each plane had intersection of three blocks with exactly seven planes in the other class. Since there are thirty total planes, and each plane is disjoint from eight planes, has intersection of one with fourteen planes, and has intersection of size three with seven planes, it follows that no two planes can have intersection of size two. To double check this, a graph was defined with edges corresponding to planes that have intersection of exactly two. As expected, this graph contained no edges.

In order to prove the properties of planes in a combinatorial fashion, the following lemma is needed:

**Lemma 5** *There are exactly three planes in Class A and three planes in Class B that contain a given block* $\{a, b, c\}$*,*

**Proof:** Let $A_1$ be a Fano plane given by:

$$A_1 = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, e, f\}, \{c, d, g\}\}$$

In any plane containing the block $\{a, b, c\}$, the points $a$ and $d$ must appear in a block together. That block can be completed in three ways, with either the point $e$, $f$, or $g$. That choice uniquely determines the other block containing $a$. There also must be a block with $b$ and $d$, which can be completed in two ways once the blocks containing $a$ are determined. Once this block is set, the remainder of the plane is determined uniquely, giving a total of six different planes that contain the

block $\{a, b, c\}$. The other five planes containing $\{a, b, c\}$ are as follows:

$$A_2 = \{\{a, b, c\}, \{a, d, f\}, \{a, e, g\}, \{b, d, g\}, \{b, e, f\}, \{c, d, e\}, \{c, f, g\}\}$$
$$A_3 = \{\{a, b, c\}, \{a, d, g\}, \{a, e, f\}, \{b, d, e\}, \{b, f, g\}, \{c, d, f\}, \{c, e, g\}\}$$
$$A_4 = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, g\}, \{b, e, f\}, \{c, e, g\}, \{c, d, f\}\}$$
$$A_5 = \{\{a, b, c\}, \{a, d, f\}, \{a, e, g\}, \{b, d, e\}, \{b, f, g\}, \{c, d, g\}, \{c, e, f\}\}$$
$$A_6 = \{\{a, b, c\}, \{a, d, g\}, \{a, e, f\}, \{b, d, f\}, \{b, e, g\}, \{c, d, e\}, \{c, f, g\}\}$$

It is easy to check that $|A_1 \cap A_2| = 1$, $|A_1 \cap A_3| = 1$, $|A_1 \cap A_4| = 3$, $|A_1 \cap A_5| = 3$, and $|A_1 \cap A_6| = 3$. Therefore, $A_1$, $A_2$, and $A_3$ are in one class and $A_4$, $A_5$, and $A_6$ are in the other. ■

Remember that the two classes were defined by two planes are in the same class if they share exactly one block together. We now present a theorem concerning the structure of the two classes of Fano planes.

**Theorem 6** *The following are properties of the two classes of Fano planes:*

*1.) Each class contains 15 of the 30 total planes.*

*2.) Given two planes in different classes, they are either disjoint or their intersection contains three blocks.*

*3.) Each plane is disjoint from 8 planes in the other class and has intersection of 3 with 7 planes in the other class.*

**Proof:**

1.) By Lemma 5, we have that there are exactly three planes in class $A$ and three planes in class $B$ that contain a given block $\{a, b, c\}$. Therefore, given a plane $A_1$, there are fourteen planes that have intersection of exactly one with $A_1$ (two planes for every block in $A_1$.) This gives a total of fifteen planes in the class containing $A_1$.

2.) Assume that we have two planes that intersect in more than one block, i.e. there are at least two blocks that the planes have in common. By definition every two blocks in a Fano plane intersect in exactly one point, therefore the two blocks

that the planes intersect in must have a point in common. The third block containing that point is uniquely determined since it must appear in a block with the other two points not yet used. Since four blocks uniquely determine a plane, the two planes either intersect in three or are identical. This shows that if two planes intersect in more than one block, they either intersect in three blocks or are identical planes.

3.) Once again using Lemma 5, given a block in the plane $A_1$, there are exactly three planes in class $B$ that contain that particular block. There are seven blocks in plane $A_1$, giving twenty-one occurrences of blocks of $A_1$ appearing in class $B$. Any plane in class $B$ that has non-empty intersection with $A_1$ must intersect in exactly three blocks. Therefore, there must be seven planes in class $B$ that intersect with $A_1$ to account for the twenty-one block occurrences. Since there are fifteen planes in class $B$ and every plane $B_i$ either has intersection of three or is disjoint from a given $A_i$, it follows that each $A_i$ is disjoint from exactly eight planes. ■

One further property of these classes of Fano planes is considered in Section 2.4.

## 2.3 Sets of Fano Planes with Maximum Distinct Blocks

Since we determined that it is not possible to get five Room squares of side seven containing all thirty-five different row patterns, the next question we asked was given five Room squares of side seven, what is the *maximum* number $x$ of distinct row patterns that can occur. Clearly, $x \leq 34$. The cases of two, three, four, and six Room squares were also examined. Since having a Room square of side seven is equivalent to having a Fano plane, the below results are discussed in terms of Fano planes with each distinct block corresponding to a distinct zero-row pattern in the corresponding Room square.

Results were first computed using the *Mathematica* software package, see Ap-

pendix B for code. It was determined that a set of two Fano planes can contain at most fourteen distinct blocks, a set of three can contain twenty distinct blocks, a set of four can contain twenty-six distinct blocks, a set of five can contain thirty-one distinct blocks, and all thirty-five possible blocks can be placed in six Fano planes. We then used the properties of the two isomorphism classes of Fano planes to determine that for each of the aforementioned sets of planes, there is only one possible structure. The following two lemmas will be used extensively in the proofs which follow.

**Lemma 7** *If three planes from one class all intersect in the same block, there does not exist a plane from the other class that is disjoint from all three.*

**Proof:** Let $A_1$, $A_2$, and $A_3$ be three planes from class $A$, with $\{a, b, c\}$ be the block they share in common. We can say without loss of generality that $A_1$ contains $\{a, d, e\}$ and $\{a, f, g\}$, $A_2$ contains $\{a, d, f\}$ and $\{a, e, g\}$, and $A_3$ contains $\{a, d, g\}$ and $\{a, e, f\}$ because the point $a$ must occur with every other point exactly once in a Fano plane and any other arrangement would either cause one of $A_1$, $A_2$, or $A_3$ to not be a Fano plane or would cause two of the planes to intersect in more than one block (which would contradict them coming from the same class).

For a plane $B_1$ to be disjoint from $A_1$, $A_2$, and $A_3$, it must contain either $\{a, b, d\}$ or $\{a, c, d\}$ since $a$ and $d$ must appear in a block together and the third element in the block cannot be $e$, $f$, or $g$ since those blocks occur in one of the $A_i$. Likewise, $B_1$ must contain either $\{a, b, e\}$ or $\{a, c, e\}$ and either $\{a, b, g\}$ or $\{a, c, g\}$. There is no way to satisfy all three of those requirements without either having $a$ and $b$ or $a$ and $c$ in two separate blocks, which contradicts the structure of the Fano plane. Therefore, there cannot be a plane from class $B$ that is disjoint from $A_1$, $A_2$, and $A_3$.

By the symmetry of the classes, the proof is the same if $A_1$, $A_2$, and $A_3$ are all in Class $B$. ■

**Lemma 8** *Given any pair of planes, $A_1$ and $A_2$ from one class there are at least four planes from the other class that are disjoint from both $A_1$ and $A_2$.*

**Proof:** Without loss of generality, let $A_1$ and $A_2$ be a pair of planes from class $A$. Let $a$ be the block that $A_1$ and $A_2$ have in common. There are three planes in class $B$ that contain $a$ by Lemma 5. These three planes cannot be disjoint from $A_1$ or $A_2$. This leaves twelve planes that could be disjoint from one or both of $A_1$ and $A_2$. Each $A_i$ is disjoint from eight $B_i$ by Theorem 6. Placing two sets of eight onto a set of twelve guarantees an overlap of at least four by the pigeonhole principle. Therefore, $A_1$ and $A_2$ must have at least four planes disjoint from both of them. ■

The next question addressed is how many distinct sets of two, three, four, five and six Fano planes with the maximum number of distinct blocks there were and within those groups, how many isomorphism classes were present. The total number of distinct sets of Fano planes with the maximum number of different blocks was determined using *Mathematica*. Using the computer software, each possible set of the given number of planes was checked to see if it contained the desired number of distinct blocks. The sets that had the maximum number of distinct blocks were then appended to a list that contained all of the sets with that property. The next step was to check for the number of isomorphism classes.

Before determining the number of isomorphism classes for each group, we generated and stored a list of all the permutations on seven elements. The permutations were numbered in lexicographic order from 1 through 7!. For a given plane $A_i$, we let each permutation $k$ act on $A_i$. The lists $\sigma_{i,j}$ kept track of the permutations $k$ that mapped plane $A_i$ to plane $A_j$. Given two different sets of Fano planes with the maximum number of distinct blocks, we determined the ways to map one set to the other that preserved the necessary structure. For example, given two sets of three

Fano planes containing 20 blocks, the structure dictates that each set contains two planes from one class and one plane from the other. Let the planes in the first set be denoted $A_1$, $A_2$, and $B_1$ where $A_1$ and $A_2$ are in the same class and $B_1$ is in the other. Likewise, let the planes in the second set be denoted $C_1$, $C_2$, and $D_1$ with $C_1$ and $C_2$ being the planes from the same class. There are two ways to map set 1 to set 2 and retain the structure. In both cases $B_1$ must be mapped to $D_1$. Then $A_1$ can be mapped to either $C_1$ or $C_2$ with $A_2$ being mapped to the other. For a given mapping, if the intersection of the lists $\sigma_{i,j}$ that corresponded to the individual plane mappings is non-empty, then there was at least one permutation that mapped the first set to the second set. The existence of such a permutation implies that the two sets were isomorphic to each other. The *Mathematica* code that computed the isomorphism classes can be found in Appendix C. The results are summarized in the following sections.

### 2.3.1 Two Planes

**Theorem 9** *A set of two Fano planes contains at most fourteen distinct blocks. Furthermore, the largest number of Fano planes in which all the blocks are distinct is two.*

**Proof:** Choose a plane from Class $A$ and denote it $A_1$. By Theorem 6, there are eight planes in Class $B$ that are disjoint from $A_1$. Therefore, $A_1$ and any one of these eight planes from Class $B$ will contain fourteen distinct blocks.

Furthermore, given any set of three planes, at least two must be from the same class by the pigeonhole principle. The two planes from the same class have non-empty intersection. Therefore any set of three cannot contain 21 distinct blocks. ■

There are $15 * 8 = 120$ different pairs of Fano planes that contain fourteen different blocks. There is only one isomorphism class for these pairs, meaning that

any pair can be transformed into any other pair with one permutation. The smallest lexicographic set of two disjoint planes are below. Note, that in this pair and in each of the subsequent sets, the planes labeled $A_i$ are from one class and the planes labeled $B_i$ are from the other class.

$$A_1 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$
$$B_1 = \{\{1,2,4\},\{1,3,6\},\{1,5,7\},\{2,3,7\},\{2,5,6\},\{3,4,5\},\{4,6,7\}\}$$

### 2.3.2 Three Planes

**Theorem 10** *A set of three Fano planes contains at most twenty distinct blocks, and any set of three Fano planes with twenty distinct blocks must contain two planes from one class and one from the other.*

**Proof:** Observe that any set of three planes must contain two planes from the same class by the pigeonhole principle. Therefore, there cannot be a set of three planes that contains 21 distinct blocks.

Choose two planes from Class $A$, call them $A_1$ and $A_2$. By definition, these planes intersect in exactly one block. Therefore, they contain thirteen distinct blocks. By Lemma 8 there exists a plane in class $B$ which is disjoint from both $A_1$ and $A_2$, call it $B_1$. Adding $B_1$ to the set with $A_1$ and $A_2$ adds seven distinct new blocks, for a total of twenty distinct blocks.

One question that arises is whether or not the plane $B_1$ has to be disjoint from both $A_1$ and $A_2$. Assume that $B_1$ is not disjoint from $A_1$. Then $B_1$ intersects with $A_1$ in exactly three blocks. Therefore, adding $B_1$ to the set $A_1$ and $A_2$ adds at most four new blocks (less if $B_1$ also has non-empty intersection with $A_2$). This gives at most seventeen distinct blocks.

We know that it is possible to achieve twenty distinct blocks with two planes from class $A$ and one from class $B$, but is it possible to achieve the same thing with

a different structure? The only other way to choose three planes is to have all three come from the same class. Any pair of planes from the same class intersect in exactly one block, giving thirteen distinct blocks for any pair. Let $A_1$ and $A_2$ be the first two planes in our set and let $a$ be the block in which they intersect. If $A_3$ contains block $a$, then it adds six new blocks for a total of nineteen. If $A_3$ does not contain block $a$ then it must intersect $A_1$ and $A_2$ in separate blocks, causing it to add five new blocks to the set for a total of eighteen. ■

There are 840 different sets of three of Fano planes that contain twenty distinct blocks. As with the pairs, it was checked on the computer that there is only one isomorphism class of these triples, with the smallest lexicographic set as follows:

$$A_1 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$
$$A_2 = \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\},\{2,5,6\},\{3,4,5\},\{3,6,7\}\}$$
$$B_1 = \{\{1,2,4\},\{1,3,7\},\{1,5,6\},\{2,3,5\},\{2,6,7\},\{3,4,6\},\{4,5,7\}\}$$

### 2.3.3 Four Planes

**Theorem 11** *A set of four Fano planes containing the maximum number of distinct blocks contains twenty-six distinct blocks and must contain two planes from Class A and two planes from Class B.*

**Proof:** Choose two planes from Class $A$ and call them $A_1$ and $A_2$. By Lemma 8 there are at least four $B_i$ that are disjoint from both $A_1$ and $A_2$. Choose $B_1$ and $B_2$ from the set of $B_i$ that are disjoint from both $A_1$ and $A_2$. The pair $A_1$ and $A_2$ and the pair $B_1$ and $B_2$ each contain thirteen distinct blocks, giving twenty-six distinct blocks in the set $A_1, A_2, B_1$ and $B_2$.

Obviously, if one of the $B_i$ was not disjoint from one of the $A_i$, a total of twenty-six distinct blocks would not be attainable. But, is it possible to attain twenty-six

with a different structure? The only other possible ways to choose four planes is to have three from Class $A$ and one from Class $B$ or to have all four come from the same class. Assume we have at least three blocks from Class $A$. If $A_1$, $A_2$, and $A_3$ have unique pairwise intersections, then $|A_1 \cup A_2 \cup A_3| = 7+6+5 = 18$. Adding one more plane, whether it be from Class $A$ or Class $B$ can add at most seven new blocks for a total of twenty-five. So, assume that $A_1$, $A_2$, and $A_3$ intersect in the same block, then $|A_1 \cup A_2 \cup A_3| = 19$. In order to attain twenty-six distinct blocks, the plane we add must add seven new blocks. Therefore, the new plane cannot come from Class $A$. However, Lemma 7 states that there is no plane from Class $B$ that is disjoint from $A_1$, $A_2$, and $A_3$. Therefore, we cannot have three planes in the same class and still obtain twenty-six distinct blocks. This eliminates all possible structures except for two planes from Class $A$ and two planes from Class $B$. ■

There are 630 different sets of four Fano planes that contain twenty-six distinct blocks. Again, there is only one isomorphism class within these sets, and the smallest lexicographic set is as follows:

$$A_1 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$
$$A_2 = \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\},\{2,5,6\},\{3,4,5\},\{3,6,7\}\}$$
$$B_1 = \{\{1,2,4\},\{1,3,7\},\{1,5,6\},\{2,3,5\},\{2,6,7\},\{3,4,6\},\{4,5,7\}\}$$
$$B_2 = \{\{1,2,5\},\{1,3,6\},\{1,4,7\},\{2,3,4\},\{2,6,7\},\{3,5,7\},\{4,5,6\}\}$$

### 2.3.4 Five Planes

**Theorem 12** *A set of five Fano planes containing the maximum number of distinct blocks contains thirty-one distinct blocks and must contain three planes from one class and two planes from the other class.*

**Proof:** We can assume without loss of generality that in our set of five planes, there are three planes from class $A$. Assume that $A_1, A_2$, and $A_3$ intersect in the same

block, yielding nineteen distinct blocks among the three of them. By Lemma 7 there does not exist any $B_i$ that is disjoint from all three $A_i$. Therefore, adding two planes $B_1$ and $B_2$ add at most four new blocks each, for a total of only twenty-seven blocks. Therefore, $A_1, A_2$, and $A_3$ must have unique pairwise intersections. Choose $B_1$ and $B_2$ such that each one is disjoint from all three $A_i$, (such a pair exists by Lemma 8). This gives eighteen distinct blocks in the $A_i$ and thirteen in the $B_i$, for a total of thirty-one distinct blocks.

The other possible structures are four planes from one class and one from the other class or all five planes from the same class. Assume we have four planes from Class $A$. There are two possible cases. Case one consists of all of the intersections being distinct, while case two contains a set of three that intersect in the same block.

Looking at case one, $A_1$ and $A_2$ contain thirteen blocks, adding $A_3$ adds five new blocks and adding $A_4$ adds four new blocks. This gives a total of twenty-two distinct blocks. Adding a fifth plane, whether it be in Class $A$ or Class $B$ can add at most seven new blocks, which gives a total of twenty-nine. Therefore, there cannot be four planes in the same class with all of the pairwise intersections being distinct in a set containing the maximum number of blocks.

Case two starts with $A_1$, $A_2$, and $A_3$ having the same block in common, for a total of nineteen distinct blocks. Lemma 5 guarantees that $A_4$ cannot have the same block in common. Therefore, adding $A_4$ to the set adds four new blocks (since it has to have non-empty intersection with each of the other $A_i$). This gives a total of twenty-three distinct blocks. Adding a fifth plane can add at most seven new blocks, giving a total of thirty blocks, which is less than the goal of thirty-one distinct blocks.

Therefore, it is not possible to obtain thirty-one distinct blocks in a set of five planes if four of them are from the same class. It follows that it is also not possible when all five planes are from the same class. ■

There are 840 different sets of five Fano planes that contain thirty-one distinct blocks. Using the computer, it was verified that there is only one isomorphism class within these sets, and the smallest lexicographic set is as follows:

$$A_1 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$
$$A_2 = \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\},\{2,5,6\},\{3,4,5\},\{3,6,7\}\}$$
$$A_3 = \{\{1,2,4\},\{1,3,5\},\{1,6,7\},\{2,3,7\},\{2,5,6\},\{3,4,6\},\{4,6,7\}\}$$
$$B_1 = \{\{1,2,5\},\{1,3,6\},\{1,4,7\},\{2,3,4\},\{2,6,7\},\{3,5,7\},\{4,5,6\}\}$$
$$B_2 = \{\{1,2,7\},\{1,3,4\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{3,5,7\},\{4,6,7\}\}$$

### 2.3.5 Six Planes

The following theorem characterizes the structure of a set of six planes containing all thirty-five possible blocks. Because of the symmetry of the classes and the symmetry present in the sets of two and four planes with the maximum number of distinct blocks, we suspected that the structure of this set of six planes would also be symmetric and contain three planes from Class $A$ and three planes from Class $B$. Surprisingly, it ended up that this is not the case. Instead, any set of six planes containing all thirty-five blocks must contain four planes from one class and two from the other.

**Theorem 13** *A set of six Fano planes containing all thirty-five possible blocks contains four planes from one class and two planes from the other class.*

**Proof:** Start with a set of five planes that have thirty-one distinct blocks which, without loss of generality, contains three planes from class $A$ and two planes from class $B$. Say this set is $A_1, A_2, A_3, B_1$, and $B_2$. By Lemma 8, we know that there are at least four $A_i$ that are disjoint from both $B_1$ and $B_2$. We also know that these $A_i$ must have different pairwise intersections by Lemma 7. Therefore, adding $A_4$ adds

exactly four new blocks, for the total of thirty-five distinct blocks, the maximum possible. So the structure with four $A_i$ with different pairwise intersections and two $B_i$ yields the desired number of blocks.

Assume that three of the $A_i$ had the same block in common. This would give nineteen unique blocks among $A_1, A_2$, and $A_3$. Adding $A_4$ adds four new blocks for a total of twenty-three blocks. By Lemma 7, we know that $|A_i \cap B_k| = 3$ for some $i$ and $k$. So adding $B_1$ and $B_2$ to the set adds at most eleven new blocks (seven from one and four from the other). This is still short of the goal of thirty-five unique blocks. Therefore, the $A_i$ must have different pairwise intersection.

Assume that we have three planes from each class and both $|A_1 \cup A_2 \cup A_3| = 18$ and $|B_1 \cup B_2 \cup B_3| = 18$. This gives a total of thirty-six blocks if all were unique, but we know we only have thirty-five possible blocks. Therefore one of the $A_i$ must have non-empty intersection with one of the $B_i$. We also know that if $|A_i \cap B_j| \neq \emptyset$ then $|A_i \cap B_j| = 3$. Therefore, adding the $B_i$ to the $A_i$ only adds fifteen new blocks, for a total of thirty-three distinct blocks.

Assume that we have three planes from each class and either $|A_1 \cup A_2 \cup A_3| = 19$ or $|B_1 \cup B_2 \cup B_3| = 19$. Assume without loss of generality that $|A_1 \cup A_2 \cup A_3| = 19$. By Lemma 7, we cannot find any $B_i$ that is disjoint from all of the $A_i$. So each $B_i$ adds at most four new blocks to the set, giving thirty-one total distinct blocks. Therefore, there are not six planes containing all thirty-five blocks that contains three planes from class $A$ and three planes from class $B$.

Assume that we have five planes from class $A$ and one plane from class $B$. If all of the pairwise intersections between the $A_i$ are distinct, we get $7+6+5+4+3 = 25$ distinct blocks. Adding $B_1$ adds at most seven new blocks, for a total of thirty-two. If we have $|A_1 \cup A_2 \cup A_3| = 19$ and every other pairwise intersection is distinct we get $7 + 6 + 6 + 4 + 3 = 26$ distinct blocks. Adding seven from a new plane still does

not give the desired thirty-five blocks. The last possibility is that $|A_1 \cup A_2 \cup A_3| = 19$ and $|A_4 \cup A_5 \cup A_i| = 19$ for $i = 1, 2$, or 3. In this case we get twenty-seven distinct blocks, to which adding seven from a new plane gives thirty-four. Therefore, it is not possible to have five of our six planes from the same class and obtain all thirty-five different blocks. Similarly, if we cannot have five from the same class, then we also cannot have six from the same class. So, the only possible structure for six planes containing all thirty-five distinct blocks is to have four from one class and two from the other. ■

There are only 210 different sets of six Fano planes that contain all thirty-five different blocks. Once again, there is only one isomorphism class of these sets. The smallest lexicographic one is below:

$$
\begin{aligned}
A_1 &= \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\} \\
A_2 &= \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\},\{2,5,6\},\{3,4,5\},\{3,6,7\}\} \\
A_3 &= \{\{1,2,4\},\{1,3,5\},\{1,6,7\},\{2,3,7\},\{2,5,6\},\{3,4,6\},\{4,6,7\}\} \\
A_4 &= \{\{1,2,6\},\{1,3,7\},\{1,4,5\},\{2,3,5\},\{2,4,7\},\{3,4,6\},\{5,6,7\}\} \\
B_1 &= \{\{1,2,5\},\{1,3,6\},\{1,4,7\},\{2,3,4\},\{2,6,7\},\{3,5,7\},\{4,5,6\}\} \\
B_2 &= \{\{1,2,7\},\{1,3,4\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{3,5,7\},\{4,6,7\}\}
\end{aligned}
$$

## 2.4 Creating A Fano Plane from 7 Fano Planes

As noted above, given a plane $A_1$, there are exactly seven $B_i$ that have intersection of three with $A_1$. Label these seven planes $B_1 - B_7$. Define a $(7, 3, 1)$-BIBD as follows: Let $X$ be the set of points with $X = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$. For notational purposes, we will let $a_i$ denote the original blocks of $A_1$ and $b_i$ denote the original blocks occurring in $B_i$. Define $D_i = \{B_j | a_i \in B_j\}$ to be the new blocks containing $B_i$ as points.

**Theorem 14** *Given any Fano plane $A$ with blocks $\{a_1, /dots, a_7\}$, the seven planes*

$\{B_1, \ldots, B_7\}$ *intersecting* $A$ *is precisely three blocks form a Fano plane* $(X, \mathcal{A})$ *where* $X = \{B_1, \ldots, B_7\}$ *and* $\mathcal{A} = \{\mathcal{D}_\infty, \ldots, \mathcal{D}_i\}$ *where* $D_i = \{B_j | a_i \in B_j\}$.

**Proof:** To prove any combinatorial structure is a $(7, 3, 1)$-BIBD it is necessary to prove the following things:

1.) There are seven points.

2.) Each block contains three points.

3.) Given any pair of points, they occur in exactly one block together.

Condition 1 is proved trivially since we defined the design to have seven points. Condition 2 is also trivial since each block of $A_1$ appears in three $B_i$ by Lemma 5. Lastly, by the way we defined the structure of the classes $A$ and $B$, any two $B_i$ intersect in exactly one $b_i$. Let $B_j$ and $B_k$ be elements of $X$. If the block they intersect in is $a_i$, a block from $A_1$, then $B_j$ and $B_k$ occur in block $D_i$ together.

The only other scenario we have to worry about is if two $B_i$ intersect in a block $b_i$ that is not one the blocks $a_i$. Let $B_1$ and $B_2$ be elements of $X$. We can assume without loss of generality that $\{a_1, a_2, a_3\}$ are the blocks of $A_1$ in $B_1$ and let $\{a_4, a_5, a_6\}$ be blocks in $B_2$. Therefore, $B_1$ and $B_2$ intersect in a block not contained in $A_1$. Lemma 5 states that $a_1, a_2$, and $a_3$ must all occur in two more $B_i$. However, they cannot occur in the same $B_i$ since they already occur together in $B_1$ and that would give an intersection of greater than one between $B_1$ and this $B_i$. But, we have six blocks $a_i$ to place and only five $B_i$ with non-empty intersection with $A_1$ left. By the pigeonhole principle, one of these must contain two of these blocks, which is a contradiction. Therefore, each $B_i$ in the set $X$ must intersect in a block $a_i$. It follows that each pair of $B_i$ must occur in a block $D_i$ together. ■

An example of a Fano plane created from Fano planes in this manner follows. In this example, $D_i$ is defined to be the block in which all of the $B_j$ in this block

contain block $a_i$, where $a_i$ is the $i^{th}$ lexicographic block of $A_1$.

$$A_1 = \{\{a,b,c\},\{a,d,e\},\{a,f,g\},\{b,d,f\},\{b,e,g\},\{c,d,g\},\{c,e,f\}\}$$

$$\begin{aligned}
B_1 &= \{\{a,b,c\},\{a,d,e\},\{a,f,g\},\{b,d,g\},\{b,e,f\},\{c,e,g\},\{c,d,f\}\} \\
B_2 &= \{\{a,b,c\},\{a,d,f\},\{a,e,g\},\{b,d,e\},\{b,f,g\},\{c,d,g\},\{c,e,f\}\} \\
B_3 &= \{\{a,b,c\},\{a,d,g\},\{a,e,f\},\{b,d,f\},\{b,e,g\},\{c,d,e\},\{c,f,g\}\} \\
B_4 &= \{\{a,b,d\},\{a,c,e\},\{a,f,g\},\{b,c,f\},\{b,e,g\},\{c,d,g\},\{d,e,f\}\} \\
B_5 &= \{\{a,b,e\},\{a,c,d\},\{a,f,g\},\{b,c,g\},\{b,d,f\},\{c,e,f\},\{d,e,g\}\} \\
B_6 &= \{\{a,b,f\},\{a,c,g\},\{a,d,e\},\{b,c,d\},\{b,e,g\},\{c,e,f\},\{d,f,g\}\} \\
B_7 &= \{\{a,b,g\},\{a,c,f\},\{a,d,e\},\{b,c,e\},\{b,d,f\},\{c,d,g\},\{e,f,g\}\}
\end{aligned}$$

$$\begin{aligned}
D_1 &= \{B_1, B_2, B_3\} \\
D_2 &= \{B_1, B_6, B_7\} \\
D_3 &= \{B_1, B_4, B_5\} \\
D_4 &= \{B_3, B_5, B_7\} \\
D_5 &= \{B_3, B_4, B_6\} \\
D_6 &= \{B_2, B_4, B_7\} \\
D_7 &= \{B_2, B_5, B_6\}
\end{aligned}$$

# 3 Solution for Room Squares of Side 11

## 3.1 Hadamard Room Squares of Side 11

We define a Hadamard Room Square to be a Room Square whose pattern or zero-pattern is the incidence matrix of a Hadamard design. Previously it was shown that the zero-pattern for every Room square of size 7 is equivalent to a $(7, 3, 1)$-BIBD. Therefore, every Room square of side seven is a Hadamard Room square. 11 is a much bigger case, and unlike the Room squares of side 7, there is no list of all possible non-equivalent Room squares of side 11. It is surely not the case that all Room squares of side 11 are Hadamard Room squares, but we show below that a Hadamard Room square of side 11 exists.

A Hadamard Room square of side 11 can be constructed by using the fill pattern

created by filling the quadratic non-residue positions in $GF(11)$ of the first row. The cells can then be filled using the starter-adder method. Below is a Hadamard Room square created using the starters $S = \{\{1,2\},\{7,9\},\{3,6\},\{4,8\},\{5,10\}\}$ and $P = \{\{5,6\},\{10,1\},\{4,7\},\{9,2\},\{3,8\}\}$ and adder $A = \{4,3,1,5,9\}$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0∞ | | 5 10 | | | | 4 8 | 1 2 | 7 9 | | 3 6 |
| 4 7 | 1∞ | | 6 0 | | | | 5 9 | 2 3 | 8 10 | |
| | 5 8 | 2∞ | | 7 1 | | | | 6 10 | 3 4 | 9 0 |
| 10 1 | | 6 9 | 3∞ | | 8 2 | | | | 7 0 | 4 5 |
| 5 6 | 0 2 | | 7 10 | 4∞ | | 9 3 | | | | 8 1 |
| 9 2 | 6 7 | 1 3 | | 8 0 | 5∞ | | 10 4 | | | |
| | 10 3 | 7 8 | 2 4 | | 9 1 | 6∞ | | 0 5 | | |
| | | 0 4 | 8 9 | 3 5 | | 10 2 | 7∞ | | 1 6 | |
| | | | 1 5 | 9 10 | 4 6 | | 0 3 | 8∞ | | 2 7 |
| 3 8 | | | | 2 6 | 10 0 | 5 7 | | 1 4 | 9∞ | |
| | 4 9 | | | | 3 7 | 0 1 | 6 8 | | 2 5 | 10∞ |

The fill pattern for this square is the incidence matrix of an $(11,6,3)$-BIBD. Likewise, we note that the zero-pattern for this square is the incidence matrix of an $(11,5,2)$-BIBD. There is only one $(11,5,2)$ design up to isomorphism, so the above room square can be changed by row and column permutations so that the zero-pattern is the incidence matrix of any $(11,5,2)$ design. Hence, if we can find a large set of 42 $(11,5,2)$ BIBDs, we can present 42 RS(11) whose $42 * 11 = 462$ row patterns are all $\binom{11}{2}$ possible fill patterns. We do this in the next section.

## 3.2 Large Set of (11,5,2)-BIBDs

A large set will contain 42 designs. This number can be determined by the fact that there are $\binom{11}{5}$ possible blocks and each design contains 11 blocks since $(11,5,2)$-BIBDs are symmetric. In [10], a large set was obtained by first creating six starter

designs obtained by six permutations acting on a single base design. The permutation $\sigma$= *(1)(2)(3)(4)(5 6 7 8 9 10 11)* was then used to obtain seven different $(11, 5, 2)$-BIBDs from each of the six starter designs.

Given a permutation $\pi$ and a large set $B$, let the image of $\pi$ acting on $B$ be denoted by $B^*$. A large set $B$ of $t$-designs is said to be *$G$-invariant* if any permutation $\pi \in G$, the resulting image $B^*$ is the same as $B$. When creating the large set, $\sigma$ was chosen as the permutation to use for all the systems because if we let $G =< \sigma >$, then the search for a large set was narrowed by noting that the construction above will result in a $G$-invariant large set when $G =< \sigma >$.

The result in [10] which we will use is the following theorem: Given $\sigma$= *(1)(2)(3)(4)(5 6 7 8 9 10 11)* and $G =< \sigma >$, there are five non-isomorphic $G$-invariant $LS2 - (11, 5, 2)$. This result was obtained using computer search techniques.

The theorem gives the existence of large sets of $(11, 5, 2)$-BIBDs, which in turn gives the existence of a set of 42 Room squares of side 11 in which every row pattern is used exactly once. The following large set of $(11, 5, 2)$-BIBDs was created using *Mathematica* (see Appendix D for code) and the results of [10].

Note that there is only one non-isomorphic $(11, 5, 2)$-BIBD. The following base design was used for all calculations:

$$\mathcal{D} = \{\{1,2,3,7,10\},\{1,2,6,9,11\},\{1,3,4,5,9\},\{1,4,6,7,8\},\{1,5,8,10,11\},\{2,3,4,8,11\},\{2,4,5,6,10\},\{2,5,7,8,9\},\{3,5,6,7,11\},\{3,6,8,9,10\},\{4,7,9,10,11\}\}$$

The six starter designs were obtained by having the following six permutations act on the base design $\mathcal{D}$:

A. (1 2 5 4 6 10 8 3 7 9 11)

B. (1 4 8 11 10 5 7) (2 6) (3 9)

C. (1) (2) (3) (4) (5) (6 11 7 10 8) (9)

D. (1 3 2 8 4 6) (5 7) (9) (10) (11)

E. (1) (2) (3) (4) (5) (6 8 9 7 10) (11)

F. (1 3 11) (2 9 5 7 10) (4) (6 8)

The remaining designs were obtained by having each permutation in $G$ act on each of the starter designs. The following 42 designs were the result of these calculations. Note that $A_0$ is the starter design obtained from permutation $A$ acting on the base design and for $0 \leq i \leq 6$ each $A_i$ is the design obtained from the permutation $\sigma^i$ acting on $A_0$. Likewise, the designs labelled $B_i, C_i, D_i, E_i$ and $F_i$ were labelled in the same fashion as the $A_i$. The code used to generate this large set is included in the appendix.

$$A_0 = \{\{2,5,7,8,9\},\{1,2,5,10,11\},\{2,4,6,7,11\},\{2,3,6,9,10\},\{1,2,3,4,8\},\{1,3,5,6,7\},\{4,5,6,8,10\},\{3,4,5,9,11\},\{1,4,7,9,10\},\{3,7,8,10,11\},\{1,6,8,9,11\}\}$$

$$A_1 = \{\{2,6,8,9,10\},\{1,2,5,6,11\},\{2,4,5,7,8\},\{2,3,7,10,11\},\{1,2,3,4,9\},\{1,3,6,7,8\},\{4,6,7,9,11\},\{3,4,5,6,10\},\{1,4,8,10,11\},\{3,5,8,9,11\},\{1,5,7,9,10\}\}$$

$$A_2 = \{\{2,7,9,10,11\},\{1,2,5,6,7\},\{2,4,6,8,9\},\{2,3,5,8,11\},\{1,2,3,4,10\},\{1,3,7,8,9\},\{4,5,7,8,10\},\{3,4,6,7,11\},\{1,4,5,9,11\},\{3,5,6,9,10\},\{1,6,8,10,11\}\}$$

$$A_3 = \{\{2,5,8,10,11\},\{1,2,6,7,8\},\{2,4,7,9,10\},\{2,3,5,6,9\},\{1,2,3,4,11\},\{1,3,8,9,10\},\{4,6,8,9,11\},\{3,4,5,7,8\},\{1,4,5,6,10\},\{3,6,7,10,11\},\{1,5,7,9,11\}\}$$

$$A_4 = \{\{2,5,6,9,11\},\{1,2,7,8,9\},\{2,4,8,9,10\},\{2,3,6,7,10\},\{1,2,3,4,5\},\{1,3,9,10,11\},\{4,5,7,9,10\},\{3,4,6,8,9\},\{1,4,6,7,11\},\{3,5,7,8,11\},\{1,5,6,8,10\}\}$$

$$A_5 = \{\{2,5,6,7,10\},\{1,2,8,9,10\},\{2,4,5,9,11\},\{2,3,7,8,11\},\{1,2,3,4,6\},\{1,3,5,10,11\},\{4,6,8,10,11\},\{3,4,7,9,10\},\{1,4,5,7,8\},\{3,5,6,8,9\},\{1,6,7,9,11\}\}$$

$$A_6 = \{\{2,6,7,8,11\},\{1,2,9,10,11\},\{2,4,5,6,10\},\{2,3,5,8,9\},\{1,2,3,4,7\},\{1,3,5,6,11\},\{4,5,7,9,11\},\{3,4,8,10,11\},\{1,4,6,8,9\},\{3,6,7,9,10\},\{1,5,7,8,10\}\}$$

$$B_0 = \{\{1,4,5,6,9\},\{2,3,4,6,10\},\{3,4,7,8,9\},\{1,2,4,8,11\},\{4,5,7,10,11\},\{6,8,9,10,11\},\{2,5,6,7,8\},\{1,3,6,7,11\},\{1,2,7,9,10\},\{2,3,5,9,11\},\{1,3,5,8,10\}\}$$

$$B_1 = \{\{1,4,6,7,10\},\{2,3,4,7,11\},\{3,4,8,9,10\},\{1,2,4,5,9\},\{4,5,6,8,11\},\{5,7,9,10,11\},\{2,6,7,8,9\},\{1,3,5,7,8\},\{1,2,8,10,11\},\{2,3,5,6,10\},\{1,3,6,9,11\}\}$$

$$B_2 = \{\{1,4,7,8,11\},\{2,3,4,5,8\},\{3,4,9,10,11\},\{1,2,4,6,10\},\{4,5,6,7,9\},\{5,6,8,10,11\},\{2,7,8,9,10\},\{1,3,6,8,9\},\{1,2,5,9,11\},\{2,3,6,7,11\},\{1,3,5,7,10\}\}$$

$$B_3 = \{\{1,4,5,8,9\},\{2,3,4,6,9\},\{3,4,5,10,11\},\{1,2,4,7,11\},\{4,6,7,8,10\},\{5,6,7,9,11\},\{2,8,9,10,11\},\{1,3,7,9,10\},\{1,2,5,6,10\},\{2,4,5,7,8\},\{1,3,6,8,11\}\}$$

$$B_4 = \{\{1,4,6,9,10\},\{2,3,4,7,10\},\{3,4,5,6,11\},\{1,2,4,5,8\},\{4,7,8,9,11\},\{5,6,7,8,10\},\{2,5,9,10,11\},\{1,3,8,10,11\},\{1,2,6,7,11\},\{2,3,6,8,9\},\{1,3,5,7,9\}\}$$

$$B_5 = \{\{1,4,7,10,11\},\{2,3,4,8,11\},\{3,4,5,6,7\},\{1,2,4,6,9\},\{4,5,8,9,10\},\{6,7,8,9,11\},\{2,5,6,10,11\},\{1,3,5,9,11\},\{1,2,5,7,8\},\{2,3,7,9,10\},\{1,3,6,8,10\}\}$$

$$B_6 = \{\{1,4,5,8,11\},\{2,3,4,5,9\},\{3,4,6,7,8\},\{1,2,4,7,10\},\{4,6,9,10,11\},\{5,7,8,9,10\},\{2,5,6,7,11\},\{1,3,5,6,10\},\{1,2,6,8,9\},\{2,3,8,10,11\},\{1,3,7,9,11\}\}$$

$$C_0 = \{\{1,2,3,8,10\},\{1,2,7,9,11\},\{1,3,4,5,9\},\{1,4,6,10,11\},\{1,5,6,7,8\},\{2,3,4,6,7\},\{2,4,5,8,11\},\{2,5,6,9,10\},\{3,5,7,10,11\},\{3,6,8,9,11\},\{4,7,8,9,10\}\}$$

$$C_1 = \{\{1,2,3,9,11\},\{1,2,5,8,10\},\{1,3,4,6,10\},\{1,4,5,7,11\},\{1,6,7,8,9\},\{2,3,4,7,8\},\{2,4,5,6,9\},\{2,6,7,10,11\},\{3,5,6,8,11\},\{3,5,7,9,10\},\{4,8,9,10,11\}\}$$

$$C_2 = \{\{1,2,3,5,10\},\{1,2,6,9,11\},\{1,3,4,7,11\},\{1,4,5,6,8\},\{1,7,8,9,10\},\{2,3,4,8,9\},\{2,4,6,7,10\},\{2,5,7,8,11\},\{3,5,6,7,9\},\{3,6,8,10,11\},\{4,5,9,10,11\}\}$$

$$C_3 = \{\{1,2,3,6,11\},\{1,2,5,7,10\},\{1,3,4,5,8\},\{1,4,6,7,9\},\{1,8,9,10,11\},\{2,3,4,9,10\},\{2,4,7,8,11\},\{2,5,6,8,9\},\{3,6,7,8,10\},\{3,5,7,9,11\},\{4,5,6,10,11\}\}$$

$$C_4 = \{\{1,2,3,5,7\},\{1,2,6,8,11\},\{1,3,4,6,9\},\{1,4,7,8,10\},\{1,5,9,10,11\},\{2,3,4,10,11\},\{2,4,5,8,9\},\{2,6,7,9,10\},\{3,7,8,9,11\},\{3,5,6,8,10\},\{4,5,6,7,11\}\}$$

$$C_5 = \{\{1,2,3,6,8\},\{1,2,5,7,9\},\{1,3,4,7,10\},\{1,4,8,9,11\},\{1,5,6,10,11\},\{2,3,4,5,11\},\{2,4,6,9,10\},\{2,7,8,10,11\},\{3,5,8,9,10\},\{3,6,7,9,11\},\{4,5,6,7,8\}\}$$

$$C_6 = \{\{1,2,3,7,9\},\{1,2,6,8,10\},\{1,3,4,8,11\},\{1,4,5,9,10\},\{1,5,6,7,11\},\{2,3,4,5,6\},\{2,4,7,10,11\},\{2,5,8,9,11\},\{3,6,9,10,11\},\{3,5,7,8,10\},\{4,6,7,8,9\}\}$$

$$D_0 = \{\{2,3,5,8,10\},\{1,3,8,9,11\},\{2,3,6,7,9\},\{1,3,4,5,6\},\{3,4,7,10,11\},\{2,4,6,8,11\},\{1,6,7,8,10\},\{4,5,7,8,9\},\{1,2,5,7,11\},\{1,2,4,9,10\},\{5,6,9,10,11\}\}$$

$$D_1 = \{\{2,3,6,9,11\},\{1,3,5,9,10\},\{2,3,7,8,10\},\{1,3,4,6,7\},\{3,4,5,8,11\},\{2,4,5,7,9\},\{1,7,8,9,11\},\{4,6,8,9,10\},\{1,2,5,6,8\},\{1,2,4,10,11\},\{5,6,7,10,11\}\}$$

$$D_2 = \{\{2,3,5,7,10\},\{1,3,6,10,11\},\{2,3,8,9,11\},\{1,3,4,7,8\},\{3,4,5,6,9\},\{2,4,6,8,10\},\{1,5,8,9,10\},\{4,7,9,10,11\},\{1,2,6,7,9\},\{1,2,4,5,11\},\{5,6,7,8,11\}\}$$

$$D_3 = \{\{2,3,6,8,11\},\{1,3,5,7,11\},\{2,3,5,9,10\},\{1,3,4,8,9\},\{3,4,6,7,10\},\{2,4,7,9,11\},\{1,6,9,10,11\},\{4,5,8,10,11\},\{1,2,7,8,10\},\{1,2,4,5,6\},\{5,6,7,8,9\}\}$$

$$D_4 = \{\{2,3,5,7,9\},\{1,3,5,6,8\},\{2,3,6,10,11\},\{1,3,4,9,10\},\{3,4,7,8,11\},\{2,4,5,8,10\},\{1,5,7,10,11\},\{4,5,6,9,11\},\{1,2,8,9,11\},\{1,2,4,6,7\},\{6,7,8,9,10\}\}$$

$$D_5 = \{\{2,3,6,8,10\},\{1,3,6,7,9\},\{2,3,5,7,11\},\{1,3,4,10,11\},\{3,4,5,8,9\},\{2,4,6,9,11\},\{1,5,6,8,11\},\{4,5,6,7,10\},\{1,2,5,9,10\},\{1,2,4,7,8\},\{7,8,9,10,11\}\}$$

$$D_6 = \{\{2,3,7,9,11\},\{1,3,7,8,10\},\{2,3,5,6,8\},\{1,3,4,5,11\},\{3,4,6,9,10\},\{2,4,5,7,10\},\{1,5,6,7,9\},\{4,6,7,8,11\},\{1,2,6,10,11\},\{1,2,4,8,9\},\{5,8,9,10,11\}\}$$

$$E_0 = \{\{1,2,3,6,10\},\{1,2,7,8,11\},\{1,3,4,5,7\},\{1,4,8,9,10\},\{1,5,6,9,11\},\{2,3,4,9,11\},\{2,4,5,6,8\},\{2,5,7,9,10\},\{3,5,8,10,11\},\{3,6,7,8,9\},\{4,6,7,10,11\}\}$$

$$E_1 = \{\{1,2,3,7,11\},\{1,2,5,8,9\},\{1,3,4,6,8\},\{1,4,9,10,11\},\{1,5,6,7,10\},\{2,3,4,5,10\},\{2,4,6,7,9\},\{2,6,8,10,11\},\{3,5,6,9,11\},\{3,7,8,9,10\},\{4,5,7,8,11\}\}$$

$$E_2 = \{\{1,2,3,5,8\},\{1,2,6,9,10\},\{1,3,4,7,9\},\{1,4,5,10,11\},\{1,6,7,8,11\},\{2,3,4,6,11\},\{2,4,7,8,10\},\{2,5,7,9,11\},\{3,5,6,7,10\},\{3,8,9,10,11\},\{4,5,6,8,9\}\}$$

$$E_3 = \{\{1,2,3,6,9\},\{1,2,7,10,11\},\{1,3,4,8,10\},\{1,4,5,6,11\},\{1,5,7,8,9\},\{2,3,4,5,7\},\{2,4,8,9,11\},\{2,5,6,8,10\},\{3,6,7,8,11\},\{3,5,9,10,11\},\{4,6,7,9,10\}\}$$

$$E_4 = \{\{1,2,3,7,10\},\{1,2,5,8,11\},\{1,3,4,9,11\},\{1,4,5,6,7\},\{1,6,8,9,10\},\{2,3,4,6,8\},\{2,4,5,9,10\},\{2,6,7,9,11\},\{3,5,7,8,9\},\{3,5,6,10,11\},\{4,7,8,10,11\}\}$$

$$E_5 = \{\{1,2,3,8,11\},\{1,2,5,6,9\},\{1,3,4,5,10\},\{1,4,6,7,8\},\{1,7,9,10,11\},\{2,3,4,7,9\},\{2,4,6,10,11\},\{2,5,7,8,10\},\{3,6,8,9,10\},\{3,5,6,7,11\},\{4,5,8,9,11\}\}$$

$$E_6 = \{\{1,2,3,5,9\},\{1,2,6,7,10\},\{1,3,4,6,11\},\{1,4,7,8,9\},\{1,5,8,10,11\},\{2,3,4,8,10\},\{2,4,5,7,11\},\{2,6,8,9,11\},\{3,7,9,10,11\},\{3,5,6,7,8\},\{4,5,6,9,10\}\}$$

$$F_0 = \{\{2,3,9,10,11\},\{1,3,5,8,9\},\{3,4,5,7,11\},\{3,4,6,8,10\},\{1,2,3,6,7\},\{1,4,6,9,11\},\{2,4,7,8,9\},\{5,6,7,9,10\},\{1,7,8,10,11\},\{2,5,6,8,11\},\{1,2,4,5,10\}\}$$

$$F_1 = \{\{2,3,5,10,11\},\{1,3,6,9,10\},\{3,4,5,6,8\},\{3,4,7,9,11\},\{1,2,3,7,8\},\{1,4,5,7,10\},\{2,4,8,9,10\},\{6,7,8,10,11\},\{1,5,8,9,11\},\{2,5,6,7,9\},\{1,2,4,6,11\}\}$$

$$F_2 = \{\{2,3,5,6,11\},\{1,3,7,10,11\},\{3,4,6,7,9\},\{3,4,5,8,10\},\{1,2,3,8,9\},\{1,4,6,8,11\},\{2,4,9,10,11\},\{5,7,8,9,11\},\{1,5,6,9,10\},\{2,6,7,8,10\},\{1,2,4,5,7\}\}$$

$$F_3 = \{\{2,3,5,6,7\},\{1,3,5,8,11\},\{3,4,7,8,10\},\{3,4,6,9,11\},\{1,2,3,9,10\},\{1,4,5,7,9\},\{2,4,5,10,11\},\{5,6,8,9,10\},\{1,6,7,10,11\},\{2,7,8,9,11\},\{1,2,4,6,8\}\}$$

$$F_4 = \{\{2,3,6,7,8\},\{1,3,5,6,9\},\{3,4,8,9,11\},\{3,4,5,7,10\},\{1,2,3,10,11\},\{1,4,6,8,10\},\{2,4,5,6,11\},\{6,7,9,10,11\},\{1,5,7,8,11\},\{2,5,8,9,10\},\{1,2,4,7,9\}\}$$

$$F_5 = \{\{2,3,7,8,9\},\{1,3,6,7,10\},\{3,4,5,9,10\},\{3,4,6,8,11\},\{1,2,3,5,11\},\{1,4,7,9,11\},\{2,4,5,6,7\},\{5,7,8,10,11\},\{1,5,6,8,9\},\{2,6,9,10,11\},\{1,2,4,8,10\}\}$$

$$F_6 = \{\{2,3,8,9,10\},\{1,3,7,8,11\},\{3,4,6,10,11\},\{3,4,5,7,9\},\{1,2,3,5,6\},\{1,4,5,8,10\},\{2,4,6,7,8\},\{5,6,8,9,11\},\{1,6,7,9,10\},\{2,5,7,10,11\},\{1,2,4,9,11\}\}$$

## 3.3 42 Room Squares of Side 11 with Distinct Row Patterns

Recall the Hadamard Room square of side 11 constructed using the orthogonal starters. A simple permutation of the rows results in a Room square in which the pattern of empty cells of row $d_i$ matches the block $i$ of the starter design $\mathcal{D}$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 b | | 5 a | | | | 4 8 | 1 2 | 7 9 | | 3 6 | |
| 4 7 | 1 b | | 6 0 | | | | 5 9 | 2 3 | 8 a | | |
| | 5 8 | 2 b | | 7 1 | | | | 6 a | 3 4 | 9 0 | |
| a 1 | | 6 9 | 3 b | | 8 2 | | | | 7 0 | 4 5 | |
| 5 6 | 0 2 | | 7 a | 4 b | | 9 3 | | | | 8 1 | |
| 9 2 | 6 7 | 1 3 | | 8 0 | 5 b | | a 4 | | | | |
| | a 3 | 7 8 | 2 4 | | 9 1 | 6 b | | 0 5 | | | |
| | | 0 4 | 8 9 | 3 5 | | a 2 | 7 b | | 1 6 | | |
| | | | 1 5 | 9 a | 4 6 | | 0 3 | 8 b | | 2 7 | |
| 3 8 | | | | 2 6 | a 0 | 5 7 | | 1 4 | 9 b | | |
| | 4 9 | | | | 3 7 | 0 1 | 6 8 | | 2 5 | a b | |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 9 a | 4 6 | | 0 3 | 8 b | | 2 7 | |
| | | 0 4 | 8 9 | 3 5 | | a 2 | 7 b | | 1 6 | | |
| | 4 9 | | | | 3 7 | 0 1 | 6 8 | | 2 5 | a b | |
| | 5 8 | 2 b | | 7 1 | | | | 6 a | 3 4 | 9 0 | |
| | a 3 | 7 8 | 2 4 | | 9 1 | 6 b | | 0 5 | | | |
| 3 8 | | | | 2 6 | a 0 | 5 7 | | 1 4 | 9 b | | |
| 0 b | | 5 a | | | | 4 8 | 1 2 | 7 9 | | 3 6 | |
| a 1 | | 6 9 | 3 b | | 8 2 | | | | 7 0 | 4 5 | |
| 4 7 | 1 b | | 6 0 | | | | 5 9 | 2 3 | 8 a | | |
| 5 6 | 0 2 | | 7 a | 4 b | | 9 3 | | | | 8 1 | |
| 9 2 | 6 7 | 1 3 | | 8 0 | 5 b | | a 4 | | | | |

The large set of 42 Room squares can be created through column permutations of the Room square $D$, and can be found below. The squares are labeled according to the design from the large set in the previous section that corresponds to the locations of the empty cells. Also note that the zero-pattern in row $i$ of each of the squares is the $i^{th}$ block of the corresponding design.

$A_0$

| 2 7 |  | 0 3 | 9 a |  | 1 5 |  |  |  | 4 6 | 8 b |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 |  | 8 9 | 0 4 | 1 6 | a 2 |  |  |
| a b |  | 6 8 |  | 4 9 |  |  | 2 5 | 0 1 | 3 7 |  |
| 9 0 |  |  | 7 1 | 5 8 |  | 2 b | 3 4 |  |  | 6 a |
|  |  |  |  | a 3 | 2 4 | 7 8 |  | 6 b | 9 1 | 0 5 |
|  | 3 8 |  | 2 6 |  |  |  | 9 b | 5 7 | a 0 | 1 4 |
| 3 6 | 0 b | 1 2 |  |  |  | 5 a |  | 4 8 |  | 7 9 |
| 4 5 | a 1 |  |  |  | 3 b | 6 9 | 7 0 |  | 8 2 |  |
|  | 4 7 | 5 9 |  | 1 b | 6 0 |  | 8 a |  |  | 2 3 |
| 8 1 | 5 6 |  | 4 b | 0 2 | 7 a |  |  | 9 3 |  |  |
|  | 9 2 | a 4 | 8 0 | 6 7 |  | 1 3 |  |  | 5 b |  |

$A_1$

| 2 7 |  | 0 3 | 9 a | 8 b |  | 1 5 |  |  |  | 4 6 |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 |  |  | 8 9 | 0 4 | 1 6 | a 2 |  |
| a b |  | 6 8 |  |  | 4 9 |  |  | 2 5 | 0 1 | 3 7 |
| 9 0 |  |  | 7 1 | 6 a | 5 8 |  | 2 b | 3 4 |  |  |
|  |  |  |  | 0 5 | a 3 | 2 4 | 7 8 |  | 6 b | 9 1 |
|  | 3 8 |  | 2 6 | 1 4 |  |  |  | 9 b | 5 7 | a 0 |
| 3 6 | 0 b | 1 2 |  | 7 9 |  |  | 5 a |  | 4 8 |  |
| 4 5 | a 1 |  |  |  |  | 3 b | 6 9 | 7 0 |  | 8 2 |
|  | 4 7 | 5 9 |  | 2 3 | 1 b | 6 0 |  | 8 a |  |  |
| 8 1 | 5 6 |  | 4 b |  | 0 2 | 7 a |  |  | 9 3 |  |
|  | 9 2 | a 4 | 8 0 |  | 6 7 |  | 1 3 |  |  | 5 b |

$A_2$

| 2 7 |  | 0 3 | 9 a | 4 6 | 8 b |  | 1 5 |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 |  |  |  | 8 9 | 0 4 | 1 6 | a 2 |
| a b |  | 6 8 |  | 3 7 |  | 4 9 |  |  | 2 5 | 0 1 |
| 9 0 |  |  | 7 1 |  | 6 a | 5 8 |  | 2 b | 3 4 |  |
|  |  |  |  | 9 1 | 0 5 | a 3 | 2 4 | 7 8 |  | 6 b |
|  | 3 8 |  | 2 6 | a 0 | 1 4 |  |  |  | 9 b | 5 7 |
| 3 6 | 0 b | 1 2 |  |  | 7 9 |  |  | 5 a |  | 4 8 |
| 4 5 | a 1 |  |  | 8 2 |  |  | 3 b | 6 9 | 7 0 |  |
|  | 4 7 | 5 9 |  |  | 2 3 | 1 b | 6 0 |  | 8 a |  |
| 8 1 | 5 6 |  | 4 b |  |  | 0 2 | 7 a |  |  | 9 3 |
|  | 9 2 | a 4 | 8 0 | 5 b |  | 6 7 |  | 1 3 |  |  |

$A_3$

| 2 7 |  | 0 3 | 9 a |  | 4 6 | 8 b |  | 1 5 |  |  |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 | a 2 |  |  |  | 8 9 | 0 4 | 1 6 |
| a b |  | 6 8 |  | 0 1 | 3 7 |  | 4 9 |  |  | 2 5 |
| 9 0 |  |  | 7 1 |  |  | 6 a | 5 8 |  | 2 b | 3 4 |
|  |  |  |  | 6 b | 9 1 | 0 5 | a 3 | 2 4 | 7 8 |  |
|  | 3 8 |  | 2 6 | 5 7 | a 0 | 1 4 |  |  |  | 9 b |
| 3 6 | 0 b | 1 2 |  | 4 8 |  | 7 9 |  |  | 5 a |  |
| 4 5 | a 1 |  |  |  | 8 2 |  |  | 3 b | 6 9 | 7 0 |
|  | 4 7 | 5 9 |  |  |  | 2 3 | 1 b | 6 0 |  | 8 a |
| 8 1 | 5 6 |  | 4 b | 9 3 |  |  | 0 2 | 7 a |  |  |
|  | 9 2 | a 4 | 8 0 |  | 5 b |  | 6 7 |  | 1 3 |  |

$A_4$

| 2 7 |  | 0 3 | 9 a |  |  | 4 6 | 8 b |  | 1 5 |  |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 | 1 6 | a 2 |  |  |  | 8 9 | 0 4 |
| a b |  | 6 8 |  | 2 5 | 0 1 | 3 7 |  | 4 9 |  |  |
| 9 0 |  |  | 7 1 | 3 4 |  |  | 6 a | 5 8 |  | 2 b |
|  |  |  |  |  | 6 b | 9 1 | 0 5 | a 3 | 2 4 | 7 8 |
|  | 3 8 |  | 2 6 | 9 b | 5 7 | a 0 | 1 4 |  |  |  |
| 3 6 | 0 b | 1 2 |  |  | 4 8 |  | 7 9 |  |  | 5 a |
| 4 5 | a 1 |  |  | 7 0 |  | 8 2 |  |  | 3 b | 6 9 |
|  | 4 7 | 5 9 |  | 8 a |  |  | 2 3 | 1 b | 6 0 |  |
| 8 1 | 5 6 |  | 4 b |  | 9 3 |  |  | 0 2 | 7 a |  |
|  | 9 2 | a 4 | 8 0 |  |  | 5 b |  | 6 7 |  | 1 3 |

$A_5$

| 2 7 |  | 0 3 | 9 a |  |  |  | 4 6 | 8 b |  | 1 5 |
|---|---|---|---|---|---|---|---|---|---|---|
|  |  | 7 b | 3 5 | 0 4 | 1 6 | a 2 |  |  |  | 8 9 |
| a b |  | 6 8 |  |  | 2 5 | 0 1 | 3 7 |  | 4 9 |  |
| 9 0 |  |  | 7 1 | 2 b | 3 4 |  |  | 6 a | 5 8 |  |
|  |  |  |  | 7 8 |  | 6 b | 9 1 | 0 5 | a 3 | 2 4 |
|  | 3 8 |  | 2 6 |  | 9 b | 5 7 | a 0 | 1 4 |  |  |
| 3 6 | 0 b | 1 2 |  | 5 a |  | 4 8 |  | 7 9 |  |  |
| 4 5 | a 1 |  |  | 6 9 | 7 0 |  | 8 2 |  |  | 3 b |
|  | 4 7 | 5 9 |  |  | 8 a |  |  | 2 3 | 1 b | 6 0 |
| 8 1 | 5 6 |  | 4 b |  |  | 9 3 |  |  | 0 2 | 7 a |
|  | 9 2 | a 4 | 8 0 | 1 3 |  |  | 5 b |  | 6 7 |  |

$A_6$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 7 | | 0 3 | 9 a | 1 5 | | | | 4 6 | 8 b | | |
| | | 7 b | 3 5 | 8 9 | 0 4 | 1 6 | a 2 | | | | |
| a b | | 6 8 | | | | 2 5 | 0 1 | 3 7 | | 4 9 | |
| 9 0 | | | 7 1 | | 2 b | 3 4 | | | 6 a | 5 8 | |
| | | | | 2 4 | 7 8 | | 6 b | 9 1 | 0 5 | a 3 | |
| | 3 8 | | 2 6 | | | 9 b | 5 7 | a 0 | 1 4 | | |
| 3 6 | 0 b | 1 2 | | | 5 a | | 4 8 | | 7 9 | | |
| 4 5 | a 1 | | | 3 b | 6 9 | 7 0 | | 8 2 | | | |
| | 4 7 | 5 9 | | 6 0 | | 8 a | | | 2 3 | 1 b | |
| 8 1 | 5 6 | | 4 b | 7 a | | | 9 3 | | | 0 2 | |
| | 9 2 | a 4 | 8 0 | | 1 3 | | | 5 b | | 6 7 | |

$B_0$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | | | 9 a | 1 5 | | 2 7 | 0 3 | |
| a 2 | | | | 1 6 | | 3 5 | 8 9 | 0 4 | | 7 b | |
| 0 1 | 3 7 | | | 2 5 | 4 9 | | | | a b | 6 8 | |
| | | 6 a | | 3 4 | 5 8 | 7 1 | | 2 b | 9 0 | | |
| 6 b | 9 1 | 0 5 | | | a 3 | | 2 4 | 7 8 | | | |
| 5 7 | a 0 | 1 4 | 3 8 | 9 b | | 2 6 | | | | | |
| 4 8 | | 7 9 | 0 b | | | | | 5 a | 3 6 | 1 2 | |
| | 8 2 | | a 1 | 7 0 | | | 3 b | 6 9 | 4 5 | | |
| | | 2 3 | 4 7 | 8 a | 1 b | | 6 0 | | | 5 9 | |
| 9 3 | | | 5 6 | | 0 2 | 4 b | 7 a | | 8 1 | | |
| | 5 b | | 9 2 | | 6 7 | 8 0 | | 1 3 | | a 4 | |

$B_1$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | 0 3 | | | 9 a | 1 5 | | 2 7 | |
| a 2 | | | | 7 b | 1 6 | | 3 5 | 8 9 | 0 4 | | |
| 0 1 | 3 7 | | | 6 8 | 2 5 | 4 9 | | | | a b | |
| | | 6 a | | | 3 4 | 5 8 | 7 1 | | 2 b | 9 0 | |
| 6 b | 9 1 | 0 5 | | | | a 3 | | 2 4 | 7 8 | | |
| 5 7 | a 0 | 1 4 | 3 8 | | 9 b | | 2 6 | | | | |
| 4 8 | | 7 9 | 0 b | 1 2 | | | | | 5 a | 3 6 | |
| | 8 2 | | a 1 | | 7 0 | | | 3 b | 6 9 | 4 5 | |
| | | 2 3 | 4 7 | 5 9 | 8 a | 1 b | | 6 0 | | | |
| 9 3 | | | 5 6 | | | 0 2 | 4 b | 7 a | | 8 1 | |
| | 5 b | | 9 2 | a 4 | | 6 7 | 8 0 | | 1 3 | | |

$B_2$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | 2 7 | 0 3 | | | 9 a | 1 5 | | |
| a 2 | | | | | 7 b | 1 6 | | 3 5 | 8 9 | 0 4 | |
| 0 1 | 3 7 | | | a b | 6 8 | 2 5 | 4 9 | | | | |
| | | 6 a | | 9 0 | | 3 4 | 5 8 | 7 1 | | 2 b | |
| 6 b | 9 1 | 0 5 | | | | | a 3 | | 2 4 | 7 8 | |
| 5 7 | a 0 | 1 4 | 3 8 | | | 9 b | | 2 6 | | | |
| 4 8 | | 7 9 | 0 b | 3 6 | 1 2 | | | | | 5 a | |
| | 8 2 | | a 1 | 4 5 | | 7 0 | | | 3 b | 6 9 | |
| | | 2 3 | 4 7 | | 5 9 | 8 a | 1 b | | 6 0 | | |
| 9 3 | | | 5 6 | 8 1 | | | 0 2 | 4 b | 7 a | | |
| | 5 b | | 9 2 | | a 4 | | 6 7 | 8 0 | | 1 3 | |

$B_3$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | | 2 7 | 0 3 | | | 9 a | 1 5 | |
| a 2 | | | | 0 4 | | 7 b | 1 6 | | 3 5 | 8 9 | |
| 0 1 | 3 7 | | | | a b | 6 8 | 2 5 | 4 9 | | | |
| | | 6 a | | 2 b | 9 0 | | 3 4 | 5 8 | 7 1 | | |
| 6 b | 9 1 | 0 5 | | 7 8 | | | | a 3 | | 2 4 | |
| 5 7 | a 0 | 1 4 | 3 8 | | | | 9 b | | 2 6 | | |
| 4 8 | | 7 9 | 0 b | 5 a | 3 6 | 1 2 | | | | | |
| | 8 2 | | a 1 | 6 9 | 4 5 | | 7 0 | | | 3 b | |
| | | 2 3 | 4 7 | | | 5 9 | 8 a | 1 b | | 6 0 | |
| 9 3 | | | 5 6 | | 8 1 | | | 0 2 | 4 b | 7 a | |
| | 5 b | | 9 2 | 1 3 | | a 4 | | 6 7 | 8 0 | | |

$B_4$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | 1 5 | | 2 7 | 0 3 | | | 9 a | |
| a 2 | | | | 8 9 | 0 4 | | 7 b | 1 6 | | 3 5 | |
| 0 1 | 3 7 | | | | | a b | 6 8 | 2 5 | 4 9 | | |
| | | 6 a | | | 2 b | 9 0 | | 3 4 | 5 8 | 7 1 | |
| 6 b | 9 1 | 0 5 | | 2 4 | 7 8 | | | | a 3 | | |
| 5 7 | a 0 | 1 4 | 3 8 | | | | | 9 b | | 2 6 | |
| 4 8 | | 7 9 | 0 b | | 5 a | 3 6 | 1 2 | | | | |
| | 8 2 | | a 1 | 3 b | 6 9 | 4 5 | | 7 0 | | | |
| | | 2 3 | 4 7 | 6 0 | | | 5 9 | 8 a | 1 b | | |
| 9 3 | | | 5 6 | 7 a | | 8 1 | | | 0 2 | 4 b | |
| | 5 b | | 9 2 | | 1 3 | | a 4 | | 6 7 | 8 0 | |

$B_5$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | 9 a | 1 5 | | 2 7 | 0 3 | | | | |
| a 2 | | | | 3 5 | 8 9 | 0 4 | | 7 b | 1 6 | | | |
| 0 1 | 3 7 | | | | | | a b | 6 8 | 2 5 | 4 9 | | |
| | | 6 a | | 7 1 | | 2 b | 9 0 | | 3 4 | 5 8 | | |
| 6 b | 9 1 | 0 5 | | | 2 4 | 7 8 | | | | a 3 | | |
| 5 7 | a 0 | 1 4 | 3 8 | 2 6 | | | | | 9 b | | | |
| 4 8 | | 7 9 | 0 b | | | 5 a | 3 6 | 1 2 | | | | |
| | 8 2 | | a 1 | | 3 b | 6 9 | 4 5 | | 7 0 | | | |
| | | 2 3 | 4 7 | | 6 0 | | | 5 9 | 8 a | 1 b | | |
| 9 3 | | | 5 6 | 4 b | 7 a | | 8 1 | | | 0 2 | | |
| | 5 b | | 9 2 | 8 0 | | 1 3 | | a 4 | | 6 7 | | |

$B_6$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 6 | 8 b | | | 9 a | 1 5 | | 2 7 | 0 3 | | | |
| a 2 | | | | | 3 5 | 8 9 | 0 4 | | 7 b | 1 6 | | |
| 0 1 | 3 7 | | | 4 9 | | | | a b | 6 8 | 2 5 | | |
| | | 6 a | | 5 8 | 7 1 | | 2 b | 9 0 | | 3 4 | | |
| 6 b | 9 1 | 0 5 | | a 3 | | 2 4 | 7 8 | | | | | |
| 5 7 | a 0 | 1 4 | 3 8 | | 2 6 | | | | | 9 b | | |
| 4 8 | | 7 9 | 0 b | | | | 5 a | 3 6 | 1 2 | | | |
| | 8 2 | | a 1 | | | 3 b | 6 9 | 4 5 | | 7 0 | | |
| | | 2 3 | 4 7 | 1 b | | 6 0 | | | 5 9 | 8 a | | |
| 9 3 | | | 5 6 | 0 2 | 4 b | 7 a | | 8 1 | | | | |
| | 5 b | | 9 2 | 6 7 | 8 0 | | 1 3 | | a 4 | | | |

$C_0$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 9 a | 0 3 | 2 7 | | 8 b | | 4 6 | | |
| | | 0 4 | 8 9 | 3 5 | 7 b | | 1 6 | | a 2 | | | |
| | 4 9 | | | | 6 8 | a b | 2 5 | | 0 1 | 3 7 | | |
| | 5 8 | 2 b | | 7 1 | | 9 0 | 3 4 | 6 a | | | | |
| | a 3 | 7 8 | 2 4 | | | | | 0 5 | 6 b | 9 1 | | |
| 3 8 | | | | 2 6 | | | 9 b | 1 4 | 5 7 | a 0 | | |
| 0 b | | 5 a | | | 1 2 | 3 6 | | 7 9 | 4 8 | | | |
| a 1 | | 6 9 | 3 b | | | 4 5 | 7 0 | | | 8 2 | | |
| 4 7 | 1 b | | 6 0 | | 5 9 | | 8 a | 2 3 | | | | |
| 5 6 | 0 2 | | 7 a | 4 b | | 8 1 | | | 9 3 | | | |
| 9 2 | 6 7 | 1 3 | | 8 0 | a 4 | | | | | 5 b | | |

$C_1$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 4 6 | 9 a | 0 3 | 2 7 | | 8 b | | | |
| | | 0 4 | 8 9 | | 3 5 | 7 b | | 1 6 | | a 2 | | |
| | 4 9 | | | 3 7 | | 6 8 | a b | 2 5 | | 0 1 | | |
| | 5 8 | 2 b | | | 7 1 | | 9 0 | 3 4 | 6 a | | | |
| | a 3 | 7 8 | 2 4 | 9 1 | | | | | 0 5 | 6 b | | |
| 3 8 | | | | a 0 | 2 6 | | | 9 b | 1 4 | 5 7 | | |
| 0 b | | 5 a | | | | 1 2 | 3 6 | | 7 9 | 4 8 | | |
| a 1 | | 6 9 | 3 b | 8 2 | | | 4 5 | 7 0 | | | | |
| 4 7 | 1 b | | 6 0 | | | 5 9 | | 8 a | 2 3 | | | |
| 5 6 | 0 2 | | 7 a | | 4 b | | 8 1 | | | 9 3 | | |
| 9 2 | 6 7 | 1 3 | | 5 b | 8 0 | a 4 | | | | | | |

$C_2$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | | 4 6 | 9 a | 0 3 | 2 7 | | 8 b | | |
| | | 0 4 | 8 9 | a 2 | | 3 5 | 7 b | | 1 6 | | | |
| | 4 9 | | | 0 1 | 3 7 | | 6 8 | a b | 2 5 | | | |
| | 5 8 | 2 b | | | | 7 1 | | 9 0 | 3 4 | 6 a | | |
| | a 3 | 7 8 | 2 4 | 6 b | 9 1 | | | | | 0 5 | | |
| 3 8 | | | | 5 7 | a 0 | 2 6 | | | 9 b | 1 4 | | |
| 0 b | | 5 a | | 4 8 | | | 1 2 | 3 6 | | 7 9 | | |
| a 1 | | 6 9 | 3 b | | 8 2 | | | 4 5 | 7 0 | | | |
| 4 7 | 1 b | | 6 0 | | | | 5 9 | | 8 a | 2 3 | | |
| 5 6 | 0 2 | | 7 a | 9 3 | | 4 b | | 8 1 | | | | |
| 9 2 | 6 7 | 1 3 | | | 5 b | 8 0 | a 4 | | | | | |

$C_3$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 8 b | | 4 6 | 9 a | 0 3 | 2 7 | | | |
| | | 0 4 | 8 9 | | a 2 | | 3 5 | 7 b | | 1 6 | | |
| | 4 9 | | | | 0 1 | 3 7 | | 6 8 | a b | 2 5 | | |
| | 5 8 | 2 b | | 6 a | | | 7 1 | | 9 0 | 3 4 | | |
| | a 3 | 7 8 | 2 4 | 0 5 | 6 b | 9 1 | | | | | | |
| 3 8 | | | | 1 4 | 5 7 | a 0 | 2 6 | | | 9 b | | |
| 0 b | | 5 a | | 7 9 | 4 8 | | | 1 2 | 3 6 | | | |
| a 1 | | 6 9 | 3 b | | | 8 2 | | | 4 5 | 7 0 | | |
| 4 7 | 1 b | | 6 0 | 2 3 | | | | 5 9 | | 8 a | | |
| 5 6 | 0 2 | | 7 a | | 9 3 | | 4 b | | 8 1 | | | |
| 9 2 | 6 7 | 1 3 | | | | 5 b | 8 0 | a 4 | | | | |

$C_4$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | | 8 | b | | 4 | 6 | 9 | a | 0 | 3 | 2 7 |
| | | 0 | 4 | 8 | 9 | 1 | 6 | | a | 2 | | 3 | 5 | 7 | b |
| | 4 | 9 | | | | 2 | 5 | | 0 | 1 | 3 | 7 | | 6 | 8 a b |
| | 5 | 8 | 2 | b | | 3 | 4 | 6 | a | | | 7 | 1 | | 9 0 |
| | a | 3 | 7 | 8 | 2 | 4 | | 0 | 5 | 6 | b | 9 | 1 | | |
| 3 | 8 | | | | | 9 | b | 1 | 4 | 5 | 7 | a | 0 | 2 | 6 |
| 0 | b | | 5 | a | | | | 7 | 9 | 4 | 8 | | | 1 | 2 3 6 |
| a | 1 | | 6 | 9 | 3 | b | 7 | 0 | | | 8 | 2 | | | 4 5 |
| 4 | 7 | 1 | b | | 6 | 0 | 8 | a | 2 | 3 | | | | 5 | 9 |
| 5 | 6 | 0 | 2 | | 7 | a | | | 9 | 3 | | 4 | b | | 8 1 |
| 9 | 2 | 6 | 7 | 1 | 3 | | | | | | 5 | b | 8 | 0 | a 4 |

$C_5$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | 2 | 7 | | 8 | b | | 4 | 6 | 9 | a | 0 3 |
| | | 0 | 4 | 8 | 9 | | 1 | 6 | | a | 2 | | 3 | 5 | 7 b |
| | 4 | 9 | | | | a | b | 2 | 5 | | 0 | 1 | 3 | 7 | 6 8 |
| | 5 | 8 | 2 | b | | 9 | 0 | 3 | 4 | 6 | a | | | 7 | 1 |
| | a | 3 | 7 | 8 | 2 | 4 | | | | 0 | 5 | 6 | b | 9 | 1 |
| 3 | 8 | | | | | | 9 | b | 1 | 4 | 5 | 7 | a | 0 | 2 6 |
| 0 | b | | 5 | a | | | 3 | 6 | | 7 | 9 | 4 | 8 | | 1 2 |
| a | 1 | | 6 | 9 | 3 | b | 4 | 5 | 7 | 0 | | | 8 | 2 | |
| 4 | 7 | 1 | b | | 6 | 0 | | 8 | a | 2 | 3 | | | | 5 9 |
| 5 | 6 | 0 | 2 | | 7 | a | 8 | 1 | | | 9 | 3 | | 4 | b |
| 9 | 2 | 6 | 7 | 1 | 3 | | | | | | | | 5 | b | 8 0 a 4 |

$C_6$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | 0 | 3 | 2 | 7 | | 8 | b | | 4 | 6 | 9 a |
| | | 0 | 4 | 8 | 9 | 7 | b | | 1 | 6 | | a | 2 | | 3 5 |
| | 4 | 9 | | | | 6 | 8 | a | b | 2 | 5 | | 0 | 1 | 3 7 |
| | 5 | 8 | 2 | b | | | | 9 | 0 | 3 | 4 | 6 | a | | 7 1 |
| | a | 3 | 7 | 8 | 2 | 4 | | | | | 0 | 5 | 6 | b | 9 1 |
| 3 | 8 | | | | | | | 9 | b | 1 | 4 | 5 | 7 | a | 0 2 6 |
| 0 | b | | 5 | a | | 1 | 2 | 3 | 6 | | 7 | 9 | 4 | 8 | |
| a | 1 | | 6 | 9 | 3 | b | | 4 | 5 | 7 | 0 | | | 8 | 2 |
| 4 | 7 | 1 | b | | 6 | 0 | 5 | 9 | | 8 | a | 2 | 3 | | |
| 5 | 6 | 0 | 2 | | 7 | a | | 8 | 1 | | | 9 | 3 | | 4 b |
| 9 | 2 | 6 | 7 | 1 | 3 | | a | 4 | | | | | 5 | b | 8 0 |

$D_0$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6 | | | 0 | 3 | | 1 | 5 | 9 | a | | 8 | b | | 2 7 |
| | | 0 | 4 | | 7 | b | a | 2 | 8 | 9 | 3 | 5 | | | 1 6 |
| 3 | 7 | | | | 6 | 8 | 0 | 1 | | | | 4 | 9 | | 2 5 a b |
| | | 2 | b | | | | | | 7 | 1 | 5 | 8 | 6 | a | 3 4 9 0 |
| 9 | 1 | 7 | 8 | | | 6 | b | 2 | 4 | | a | 3 | 0 | 5 | |
| a | 0 | | | 3 | 8 | | 5 | 7 | | 2 | 6 | | 1 | 4 | 9 b |
| | | 5 | a | 0 | b | 1 | 2 | 4 | 8 | | | | 7 | 9 | |
| 8 | 2 | 6 | 9 | a | 1 | | | | 3 | b | | | | | 7 0 4 5 |
| | | | | 4 | 7 | 5 | 9 | | | 6 | 0 | | 1 | b | 2 3 8 a |
| | | | | 5 | 6 | | 8 | 1 | 9 | 3 | 7 | a | 4 | b | 0 2 |
| 5 | b | 1 | 3 | 9 | 2 | a | 4 | | | | | | 8 | 0 | 6 7 |

$D_1$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6 | | | 0 | 3 | 2 | 7 | | 1 | 5 | 9 | a | | 8 | b |
| | | 0 | 4 | | 7 | b | | a | 2 | 8 | 9 | 3 | 5 | | 1 6 |
| 3 | 7 | | | | 6 | 8 | a | b | 0 | 1 | | | 4 | 9 | 2 5 |
| | | 2 | b | | | 9 | 0 | | | 7 | 1 | 5 | 8 | 6 | a 3 4 |
| 9 | 1 | 7 | 8 | | | | | 6 | b | 2 | 4 | | a | 3 | 0 5 |
| a | 0 | | | 3 | 8 | | | 5 | 7 | | 2 | 6 | | 1 | 4 9 b |
| | | 5 | a | 0 | b | 1 | 2 | 3 | 6 | 4 | 8 | | | 7 | 9 |
| 8 | 2 | 6 | 9 | a | 1 | | | 4 | 5 | | 3 | b | | | 7 0 |
| | | | | 4 | 7 | 5 | 9 | | | 6 | 0 | | 1 | b | 2 3 8 a |
| | | | | 5 | 6 | | | 8 | 1 | 9 | 3 | 7 | a | 4 | b 0 2 |
| 5 | b | 1 | 3 | 9 | 2 | a | 4 | | | | | 8 | 0 | 6 | 7 |

$D_2$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 6 | | | 0 | 3 | | 2 | 7 | | 1 | 5 | 9 | a | | 8 b |
| | | 0 | 4 | | 7 | b | 1 | 6 | | a | 2 | 8 | 9 | 3 | 5 |
| 3 | 7 | | | | 6 | 8 | 2 | 5 | a | b | 0 | 1 | | | 4 9 |
| | | 2 | b | | | 3 | 4 | 9 | 0 | | | 7 | 1 | 5 | 8 6 a |
| 9 | 1 | 7 | 8 | | | | | | 6 | b | 2 | 4 | | a | 3 0 5 |
| a | 0 | | | 3 | 8 | | 9 | b | | 5 | 7 | | 2 | 6 | 1 4 |
| | | 5 | a | 0 | b | 1 | 2 | | 3 | 6 | 4 | 8 | | | 7 9 |
| 8 | 2 | 6 | 9 | a | 1 | | 7 | 0 | 4 | 5 | | 3 | b | | |
| | | | | 4 | 7 | 5 | 9 | 8 | a | | | 6 | 0 | | 1 b 2 3 |
| | | | | 5 | 6 | | | 8 | 1 | 9 | 3 | 7 | a | 4 | b 0 2 |
| 5 | b | 1 | 3 | 9 | 2 | a | 4 | | | | | | 8 | 0 | 6 7 |

$D_3$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 6 | | | 0 3 | 8 b | | 2 7 | | 1 5 | 9 a | |
| | 0 4 | | 7 b | | 1 6 | | a 2 | 8 9 | 3 5 | |
| 3 7 | | | 6 8 | | 2 5 | a b | 0 1 | | | 4 9 |
| | 2 b | | | 6 a | 3 4 | 9 0 | | | 7 1 | 5 8 |
| 9 1 | 7 8 | | | 0 5 | | | 6 b | 2 4 | | a 3 |
| a 0 | | 3 8 | | 1 4 | 9 b | | 5 7 | | 2 6 | |
| | 5 a | 0 b | 1 2 | 7 9 | | 3 6 | 4 8 | | | |
| 8 2 | 6 9 | a 1 | | | 7 0 | 4 5 | | 3 b | | |
| | | 4 7 | 5 9 | 2 3 | 8 a | | | 6 0 | | 1 b |
| | | 5 6 | | | | 8 1 | 9 3 | 7 a | 4 b | 0 2 |
| 5 b | 1 3 | 9 2 | a 4 | | | | | | 8 0 | 6 7 |

$D_4$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 6 | | | 0 3 | | 8 b | | 2 7 | | 1 5 | 9 a |
| | 0 4 | | 7 b | | | 1 6 | | a 2 | 8 9 | 3 5 |
| 3 7 | | | 6 8 | 4 9 | | 2 5 | a b | 0 1 | | |
| | 2 b | | | 5 8 | 6 a | 3 4 | 9 0 | | | 7 1 |
| 9 1 | 7 8 | | | a 3 | 0 5 | | | 6 b | 2 4 | |
| a 0 | | 3 8 | | | 1 4 | 9 b | | 5 7 | | 2 6 |
| | 5 a | 0 b | 1 2 | | 7 9 | | 3 6 | 4 8 | | |
| 8 2 | 6 9 | a 1 | | | | 7 0 | 4 5 | | 3 b | |
| | | 4 7 | 5 9 | 1 b | 2 3 | 8 a | | | 6 0 | |
| | | 5 6 | | 0 2 | | | 8 1 | 9 3 | 7 a | 4 b |
| 5 b | 1 3 | 9 2 | a 4 | 6 7 | | | | | | 8 0 |

$D_5$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 6 | | | 0 3 | 9 a | | 8 b | | 2 7 | | 1 5 |
| | 0 4 | | 7 b | 3 5 | | | 1 6 | | a 2 | 8 9 |
| 3 7 | | | 6 8 | | 4 9 | | 2 5 | a b | 0 1 | |
| | 2 b | | | 7 1 | 5 8 | 6 a | 3 4 | 9 0 | | |
| 9 1 | 7 8 | | | | a 3 | 0 5 | | | 6 b | 2 4 |
| a 0 | | 3 8 | | 2 6 | | 1 4 | 9 b | | 5 7 | |
| | 5 a | 0 b | 1 2 | | | 7 9 | | 3 6 | 4 8 | |
| 8 2 | 6 9 | a 1 | | | | | 7 0 | 4 5 | | 3 b |
| | | 4 7 | 5 9 | | 1 b | 2 3 | 8 a | | | 6 0 |
| | | 5 6 | | 4 b | 0 2 | | | 8 1 | 9 3 | 7 a |
| 5 b | 1 3 | 9 2 | a 4 | 8 0 | 6 7 | | | | | |

$D_6$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 6 | | | 0 3 | 1 5 | 9 a | | 8 b | | 2 7 | |
| | 0 4 | | 7 b | 8 9 | 3 5 | | | 1 6 | | a 2 |
| 3 7 | | | 6 8 | | | 4 9 | | 2 5 | a b | 0 1 |
| | 2 b | | | | 7 1 | 5 8 | 6 a | 3 4 | 9 0 | |
| 9 1 | 7 8 | | | 2 4 | | a 3 | 0 5 | | | 6 b |
| a 0 | | 3 8 | | | 2 6 | | 1 4 | 9 b | | 5 7 |
| | 5 a | 0 b | 1 2 | | | | 7 9 | | 3 6 | 4 8 |
| 8 2 | 6 9 | a 1 | | 3 b | | | | 7 0 | 4 5 | |
| | | 4 7 | 5 9 | 6 0 | | 1 b | 2 3 | 8 a | | |
| | | 5 6 | | 7 a | 4 b | 0 2 | | | 8 1 | 9 3 |
| 5 b | 1 3 | 9 2 | a 4 | | 8 0 | 6 7 | | | | |

$E_0$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 9 a | | 8 b | 4 6 | 0 3 | | 2 7 |
| | | 0 4 | 8 9 | 3 5 | 1 6 | | | 7 b | a 2 | |
| | 4 9 | | | | 2 5 | | 3 7 | 6 8 | 0 1 | a b |
| | 5 8 | 2 b | | 7 1 | 3 4 | 6 a | | | | 9 0 |
| | a 3 | 7 8 | 2 4 | | | 0 5 | 9 1 | | 6 b | |
| 3 8 | | | | 2 6 | 9 b | 1 4 | a 0 | | 5 7 | |
| 0 b | | 5 a | | | | 7 9 | | 1 2 | 4 8 | 3 6 |
| a 1 | | 6 9 | 3 b | | 7 0 | | 8 2 | | | 4 5 |
| 4 7 | 1 b | | 6 0 | | 8 a | 2 3 | | 5 9 | | |
| 5 6 | 0 2 | | 7 a | 4 b | | | | | 9 3 | 8 1 |
| 9 2 | 6 7 | 1 3 | | 8 0 | | | 5 b | a 4 | | |

$E_1$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 5 | 2 7 | 9 a | | 8 b | 4 6 | 0 3 | |
| | | 0 4 | 8 9 | | 3 5 | 1 6 | | | 7 b | a 2 |
| | 4 9 | | | a b | | 2 5 | | 3 7 | 6 8 | 0 1 |
| | 5 8 | 2 b | | 9 0 | 7 1 | 3 4 | 6 a | | | |
| | a 3 | 7 8 | 2 4 | | | | 0 5 | 9 1 | | 6 b |
| 3 8 | | | | | 2 6 | 9 b | 1 4 | a 0 | | 5 7 |
| 0 b | | 5 a | | 3 6 | | | 7 9 | | 1 2 | 4 8 |
| a 1 | | 6 9 | 3 b | 4 5 | | 7 0 | | 8 2 | | |
| 4 7 | 1 b | | 6 0 | | | 8 a | 2 3 | | 5 9 | |
| 5 6 | 0 2 | | 7 a | 8 1 | 4 b | | | | | 9 3 |
| 9 2 | 6 7 | 1 3 | | | 8 0 | | | 5 b | a 4 | |

$E_2$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | | 2 | 7 | 9 | a | | 8 | b | 4 | 6 | 0 | 3 |
| | | 0 | 4 | 8 | 9 | a | 2 | | 3 | 5 | 1 | 6 | | | 7 | b |
| | 4 | 9 | | | 0 | 1 | a | b | | 2 | 5 | | 3 | 7 | 6 | 8 |
| | 5 | 8 | 2 | b | | | 9 | 0 | 7 | 1 | 3 | 4 | 6 | a | | |
| | a | 3 | 7 | 8 | 2 | 4 | 6 | b | | | | 0 | 5 | 9 | 1 | |
| 3 | 8 | | | | | 5 | 7 | | 2 | 6 | 9 | b | 1 | 4 | a | 0 |
| 0 | b | | 5 | a | | 4 | 8 | 3 | 6 | | | 7 | 9 | | 1 | 2 |
| a | 1 | | 6 | 9 | 3 | b | | 4 | 5 | | 7 | 0 | | 8 | 2 | |
| 4 | 7 | 1 | b | | 6 | 0 | | | | | 8 | a | 2 | 3 | | 5 | 9 |
| 5 | 6 | 0 | 2 | | 7 | a | 9 | 3 | 8 | 1 | 4 | b | | | | |
| 9 | 2 | 6 | 7 | 1 | 3 | | | | 8 | 0 | | | 5 | b | a | 4 |

$E_3$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | 0 | 3 | | 2 | 7 | 9 | a | | 8 | b | 4 | 6 |
| | | 0 | 4 | 8 | 9 | 7 | b | a | 2 | | 3 | 5 | 1 | 6 | | |
| | 4 | 9 | | | 6 | 8 | 0 | 1 | a | b | | 2 | 5 | | 3 | 7 |
| | 5 | 8 | 2 | b | | | | 9 | 0 | 7 | 1 | 3 | 4 | 6 | a | |
| | a | 3 | 7 | 8 | 2 | 4 | | 6 | b | | | | 0 | 5 | 9 | 1 |
| 3 | 8 | | | | | | 5 | 7 | | 2 | 6 | 9 | b | 1 | 4 | a | 0 |
| 0 | b | | 5 | a | | 1 | 2 | 4 | 8 | 3 | 6 | | | 7 | 9 | |
| a | 1 | | 6 | 9 | 3 | b | | | 4 | 5 | | 7 | 0 | | 8 | 2 |
| 4 | 7 | 1 | b | | 6 | 0 | 5 | 9 | | | | 8 | a | 2 | 3 | |
| 5 | 6 | 0 | 2 | | 7 | a | | 9 | 3 | 8 | 1 | 4 | b | | | |
| 9 | 2 | 6 | 7 | 1 | 3 | | a | 4 | | | 8 | 0 | | | 5 | b |

$E_4$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | 4 | 6 | 0 | 3 | | 2 | 7 | 9 | a | | 8 | b |
| | | 0 | 4 | 8 | 9 | | 7 | b | a | 2 | | 3 | 5 | 1 | 6 | |
| | 4 | 9 | | | 3 | 7 | 6 | 8 | 0 | 1 | a | b | | 2 | 5 | |
| | 5 | 8 | 2 | b | | | | | 9 | 0 | 7 | 1 | 3 | 4 | 6 | a |
| | a | 3 | 7 | 8 | 2 | 4 | 9 | 1 | | 6 | b | | | | 0 | 5 |
| 3 | 8 | | | | | a | 0 | | 5 | 7 | | 2 | 6 | 9 | b | 1 | 4 |
| 0 | b | | 5 | a | | | 1 | 2 | 4 | 8 | 3 | 6 | | | 7 | 9 |
| a | 1 | | 6 | 9 | 3 | b | 8 | 2 | | | 4 | 5 | | 7 | 0 | |
| 4 | 7 | 1 | b | | 6 | 0 | | 5 | 9 | | | | | 8 | a | 2 | 3 |
| 5 | 6 | 0 | 2 | | 7 | a | | | 9 | 3 | 8 | 1 | 4 | b | | |
| 9 | 2 | 6 | 7 | 1 | 3 | | 5 | b | a | 4 | | | 8 | 0 | | |

$E_5$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | 8 | b | 4 | 6 | 0 | 3 | | 2 | 7 | 9 | a | |
| | | 0 | 4 | 8 | 9 | | | 7 | b | a | 2 | | 3 | 5 | 1 | 6 |
| | 4 | 9 | | | | 3 | 7 | 6 | 8 | 0 | 1 | a | b | | 2 | 5 |
| | 5 | 8 | 2 | b | | 6 | a | | | | 9 | 0 | 7 | 1 | 3 | 4 |
| | a | 3 | 7 | 8 | 2 | 4 | 0 | 5 | 9 | 1 | | 6 | b | | | |
| 3 | 8 | | | | | 1 | 4 | a | 0 | | 5 | 7 | | 2 | 6 | 9 | b |
| 0 | b | | 5 | a | | 7 | 9 | | 1 | 2 | 4 | 8 | 3 | 6 | | |
| a | 1 | | 6 | 9 | 3 | b | | 8 | 2 | | | 4 | 5 | | 7 | 0 |
| 4 | 7 | 1 | b | | 6 | 0 | 2 | 3 | | 5 | 9 | | | | 8 | a |
| 5 | 6 | 0 | 2 | | 7 | a | | | | 9 | 3 | 8 | 1 | 4 | b | |
| 9 | 2 | 6 | 7 | 1 | 3 | | | 5 | b | a | 4 | | | 8 | 0 | |

$E_6$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 5 | | 8 | b | 4 | 6 | 0 | 3 | | 2 | 7 | 9 | a |
| | | 0 | 4 | 8 | 9 | 1 | 6 | | | 7 | b | a | 2 | | 3 | 5 |
| | 4 | 9 | | | 2 | 5 | | 3 | 7 | 6 | 8 | 0 | 1 | a | b | |
| | 5 | 8 | 2 | b | | 3 | 4 | 6 | a | | | | 9 | 0 | 7 | 1 |
| | a | 3 | 7 | 8 | 2 | 4 | | 0 | 5 | 9 | 1 | | 6 | b | | |
| 3 | 8 | | | | | 9 | b | 1 | 4 | a | 0 | | 5 | 7 | | 2 | 6 |
| 0 | b | | 5 | a | | | | 7 | 9 | | 1 | 2 | 4 | 8 | 3 | 6 |
| a | 1 | | 6 | 9 | 3 | b | 7 | 0 | | 8 | 2 | | | 4 | 5 | |
| 4 | 7 | 1 | b | | 6 | 0 | 8 | a | 2 | 3 | | 5 | 9 | | | |
| 5 | 6 | 0 | 2 | | 7 | a | | | | | | 9 | 3 | 8 | 1 | 4 | b |
| 9 | 2 | 6 | 7 | 1 | 3 | | | | 5 | b | a | 4 | | | 8 | 0 |

$F_0$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 7 | | | 1 | 5 | 8 | b | 0 | 3 | 9 | a | 4 | 6 | | | |
| | 1 | 6 | | 8 | 9 | | | 7 | b | 3 | 5 | | | a | 2 | 0 | 4 |
| a | b | 2 | 5 | | | | | 6 | 8 | | 3 | 7 | 4 | 9 | 0 | 1 | |
| 9 | 0 | 3 | 4 | | | 6 | a | | 7 | 1 | | 5 | 8 | | 2 | b | |
| | | | | 2 | 4 | 0 | 5 | | | 9 | 1 | a | 3 | 6 | b | 7 | 8 |
| | 9 | b | 3 | 8 | | 1 | 4 | | 2 | 6 | a | 0 | | 5 | 7 | | |
| 3 | 6 | | 0 | b | | 7 | 9 | 1 | 2 | | | | | 4 | 8 | 5 | a |
| 4 | 5 | 7 | 0 | a | 1 | 3 | b | | | | | 8 | 2 | | | 6 | 9 |
| | 8 | a | 4 | 7 | 6 | 0 | 2 | 3 | 5 | 9 | | | 1 | b | | | |
| 8 | 1 | | 5 | 6 | 7 | a | | | | 4 | b | | 0 | 2 | 9 | 3 | |
| | | | 9 | 2 | | | | a | 4 | 8 | 0 | 5 | b | 6 | 7 | | 1 | 3 |

$F_1$

| 2 7 | | | 1 5 | | 8 b | 0 3 | 9 a | 4 6 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | 0 4 | | 7 b | 3 5 | | | a 2 |
| a b | 2 5 | | | | | 6 8 | | 3 7 | 4 9 | 0 1 |
| 9 0 | 3 4 | | | 2 b | 6 a | | 7 1 | | 5 8 | |
| | | | 2 4 | 7 8 | 0 5 | | | 9 1 | a 3 | 6 b |
| | 9 b | 3 8 | | | 1 4 | | 2 6 | a 0 | | 5 7 |
| 3 6 | | 0 b | | 5 a | 7 9 | 1 2 | | | | 4 8 |
| 4 5 | 7 0 | a 1 | 3 b | 6 9 | | | | 8 2 | | |
| | 8 a | 4 7 | 6 0 | | 2 3 | 5 9 | | | 1 b | |
| 8 1 | | 5 6 | 7 a | | | | 4 b | | 0 2 | 9 3 |
| | | 9 2 | | 1 3 | | a 4 | 8 0 | 5 b | 6 7 | |

$F_2$

| 2 7 | | | 1 5 | | | 8 b | 0 3 | 9 a | 4 6 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | a 2 | 0 4 | | 7 b | 3 5 | | |
| a b | 2 5 | | | 0 1 | | | 6 8 | | 3 7 | 4 9 |
| 9 0 | 3 4 | | | | 2 b | 6 a | | 7 1 | | 5 8 |
| | | | 2 4 | 6 b | 7 8 | 0 5 | | | 9 1 | a 3 |
| | 9 b | 3 8 | | 5 7 | | 1 4 | | 2 6 | a 0 | |
| 3 6 | | 0 b | | 4 8 | 5 a | 7 9 | 1 2 | | | |
| 4 5 | 7 0 | a 1 | 3 b | | 6 9 | | | | 8 2 | |
| | 8 a | 4 7 | 6 0 | | | 2 3 | 5 9 | | | 1 b |
| 8 1 | | 5 6 | 7 a | 9 3 | | | | 4 b | | 0 2 |
| | | 9 2 | | | 1 3 | | a 4 | 8 0 | 5 b | 6 7 |

$F_3$

| 2 7 | | | 1 5 | | | | 8 b | 0 3 | 9 a | 4 6 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | | a 2 | 0 4 | | 7 b | 3 5 | |
| a b | 2 5 | | | 4 9 | 0 1 | | | 6 8 | | 3 7 |
| 9 0 | 3 4 | | | 5 8 | | 2 b | 6 a | | 7 1 | |
| | | | 2 4 | a 3 | 6 b | 7 8 | 0 5 | | | 9 1 |
| | 9 b | 3 8 | | | 5 7 | | 1 4 | | 2 6 | a 0 |
| 3 6 | | 0 b | | | 4 8 | 5 a | 7 9 | 1 2 | | |
| 4 5 | 7 0 | a 1 | 3 b | | | 6 9 | | | | 8 2 |
| | 8 a | 4 7 | 6 0 | 1 b | | | 2 3 | 5 9 | | |
| 8 1 | | 5 6 | 7 a | 0 2 | 9 3 | | | | 4 b | |
| | | 9 2 | | 6 7 | | 1 3 | | a 4 | 8 0 | 5 b |

$F_4$

| 2 7 | | | 1 5 | 4 6 | | | | 8 b | 0 3 | 9 a |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | | | a 2 | 0 4 | | 7 b | 3 5 |
| a b | 2 5 | | | 3 7 | 4 9 | 0 1 | | | 6 8 | |
| 9 0 | 3 4 | | | | 5 8 | | 2 b | 6 a | | 7 1 |
| | | | 2 4 | 9 1 | a 3 | 6 b | 7 8 | 0 5 | | |
| | 9 b | 3 8 | | a 0 | | 5 7 | | 1 4 | | 2 6 |
| 3 6 | | 0 b | | | | 4 8 | 5 a | 7 9 | 1 2 | |
| 4 5 | 7 0 | a 1 | 3 b | 8 2 | | | 6 9 | | | |
| | 8 a | 4 7 | 6 0 | | 1 b | | | 2 3 | 5 9 | |
| 8 1 | | 5 6 | 7 a | | 0 2 | 9 3 | | | | 4 b |
| | | 9 2 | | 5 b | 6 7 | | 1 3 | | a 4 | 8 0 |

$F_5$

| 2 7 | | | 1 5 | 9 a | 4 6 | | | | 8 b | 0 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | 3 5 | | | a 2 | 0 4 | | 7 b |
| a b | 2 5 | | | | 3 7 | 4 9 | 0 1 | | | 6 8 |
| 9 0 | 3 4 | | | 7 1 | | 5 8 | | 2 b | 6 a | |
| | | | 2 4 | | 9 1 | a 3 | 6 b | 7 8 | 0 5 | |
| | 9 b | 3 8 | | 2 6 | a 0 | | 5 7 | | 1 4 | |
| 3 6 | | 0 b | | | | | 4 8 | 5 a | 7 9 | 1 2 |
| 4 5 | 7 0 | a 1 | 3 b | | 8 2 | | | 6 9 | | |
| | 8 a | 4 7 | 6 0 | | | 1 b | | | 2 3 | 5 9 |
| 8 1 | | 5 6 | 7 a | 4 b | | 0 2 | 9 3 | | | |
| | | 9 2 | | 8 0 | 5 b | 6 7 | | 1 3 | | a 4 |

$F_6$

| 2 7 | | | 1 5 | 0 3 | 9 a | 4 6 | | | | 8 b |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 6 | | 8 9 | 7 b | 3 5 | | | a 2 | 0 4 | |
| a b | 2 5 | | | 6 8 | | 3 7 | 4 9 | 0 1 | | |
| 9 0 | 3 4 | | | | 7 1 | | 5 8 | | 2 b | 6 a |
| | | | 2 4 | | | 9 1 | a 3 | 6 b | 7 8 | 0 5 |
| | 9 b | 3 8 | | | 2 6 | a 0 | | 5 7 | | 1 4 |
| 3 6 | | 0 b | | 1 2 | | | | 4 8 | 5 a | 7 9 |
| 4 5 | 7 0 | a 1 | 3 b | | | 8 2 | | | 6 9 | |
| | 8 a | 4 7 | 6 0 | 5 9 | | | 1 b | | | 2 3 |
| 8 1 | | 5 6 | 7 a | | 4 b | | 0 2 | 9 3 | | |
| | | 9 2 | | a 4 | 8 0 | 5 b | 6 7 | | 1 3 | |

The above 42 Room squares of side 11 provide proof of our main theorem in this section.

**Theorem 15** *There exists a set of 42 Room squares in which each of the* $\binom{11}{6}$ *possible row patterns appears exactly once.*

# Part III

# Marco Buratti's Conjecture

## 4 Background

In 2007, Marco Buratti, a professor at the Universitiá di Perusia in Italy, proposed the following conjecture: given $p$ a prime and a multiset $S$ containing $p-1$ non-zero elements from $Z_p$, there exists a Hamiltonian path in $K_p$ where the multiset of edge lengths is $S$. To understand this problem, a little background is necessary.

A *Hamiltonian Path* is a path that visits every vertex exactly once. In a given graph, a Hamiltonian path may or may not exist. However, in this problem all paths are placed within $K_p$, the complete graph on $p$ vertices. In complete graphs, Hamiltonian paths always exist. When the vertices of $K_p$ are labeled with the elements of the cyclic group $Z_p$ one can define the difference between two vertices $x$ and $y$ to be $x-y$ or $y-x$, whichever is less that $\frac{p-1}{2}$. In general, looking for a Hamiltonian path on $p$ vertices that has edge lengths equal to the multiset $S$ is equivalent to finding an ordering of the elements of $Z_p$ such that the set of differences between consecutive elements is the set $S$.

One thing to note is that when taking the difference between two consecutive elements in this ordering of $Z_p$, there is a choice between using $x_i - x_{i+1}$ in the multiset and using $x_{i+1} - x_i$. Therefore, we can limit the possible values of elements of $S$ to values between 1 and $\frac{p-1}{2}$.

# 5 Basic Cases

## 5.1 All Edges of Same Length

Let $k$ denote the length of all of the edges in the Hamiltonian path. Since $p$ is a prime and $k \leq p$, $k$ must be relatively prime to $p$. Therefore, in the group $Z_p$, the cyclic subgroup generated by $k$ contains every element. Therefore, there exists a Hamiltonian path that hits every vertex. This path can be described as follows: $\{0, k, 2k, 3k, .....(p-1)k\}$ where all elements are reduced modulo $p$. Clearly, every edge has length $k$.

## 5.2 Each Possible Length Occurring Twice

Let $S = \{1, 1, 2, 2, \ldots, \frac{p-1}{2}, \frac{p-1}{2}\}$. In this case, each possible edge length between 1 and $\frac{p-1}{2}$ occurs exactly twice. The following sequence creates a Hamiltonian path with the prescribed lengths: $(0, 1, -1, 2, -2, 3, -3, \ldots, \frac{p-1}{2}, -\frac{p-1}{2})$. It is easy to check that the sequence of differences starts with 1 and increases by 1 for each path added, giving a sequence of all non-zero elements of $Z_p$. However, it was stated earlier that for elements $x$ larger than $\frac{p-1}{2}$, the element $p - x$ can be added instead. Therefore, having a sequence of all the elements of $Z_p$ is equivalent to having the list $S = \{1, 1, 2, 2, \ldots, \frac{p-1}{2}, \frac{p-1}{2}\}$. Note that if lengths larger than $\frac{p-1}{2}$ are used, then the edges in this Hamiltonian have lengths $\{1, 2, \ldots, p\}$.

# 6 Paths with Edge Lengths a and b

The first non-trivial case we examined was that of paths where the edge lengths were from a set of exactly two values. The problem was first simplified using two basic reductions. We then found constructions that solved the reduced problem.

These reductions and constructions are discussed in the following sections.

## 6.1 General Reductions

**Lemma 16** *Let $p$ be a prime. The existence of a Hamiltonian path in $K_p$ where all edges have length $a$ or $b$ is equivalent to the existence of a Hamiltonian path where all lengths are 1 and $a^{-1}b$.*

**Proof:** Since $Z_p$ is a field, multiplicative inverses exist for all elements of $Z_p$. List the vertices of the Hamiltonian path with lengths $a$ and $b$ in the order in which they were visited, $\{x_0, x_1, \ldots, x_{p-1}\}$. Since it is a Hamiltonian path, every vertex appears exactly once. Multiply each vertex by $a^{-1}$. The resulting sequence is $\{a^{-1}x_0, a^{-1}x_1, \ldots, a^{-1}x_{p-1}\}$. Since $a^{-1}$ is relatively prime to $p$, each element of $Z_p$ appears once in the new sequence, so it is still a Hamiltonian path. Furthermore, the lengths of each path are now either $a * a^{-1} = 1$ and $a^{-1} * b$. ■

Observe that it does not matter which length was chosen to be labeled $a$ and which was chosen to be labeled $b$. Based on this, one can always label the length that occurs more often as $a$ and the length that occurs less often as $b$. (Note, if $a$ and $b$ occur equally often, either can be chosen as $a$.) This reduced the problem down to the following form:

**Lemma 17** *If a Hamiltonian path containing edges of length 1 and $k$ where the number of edges of length 1 is greater than the number of edges of length $k$ can be constructed for all $k$ such that $1 < k \leq \frac{p-1}{2}$, then there exists a Hamiltonian path with edges of length $a$ and $b$ for any number of edges of length $a$ and length $b$ (where of course the number of edges of length $a$ plus the number of edges of length $b$ is $p-1$).*

## 6.2 Constructions

Let $n$ denote the number of edges of length $k$ that we wish to place in the Hamiltonian path. Note that $1 \leq n \leq \frac{p-1}{2}$ since the second reduction implied that we only need to consider the cases in which the number of edges of length $k$ is less than the number of edges of length 1. Using the division algorithm, we have that $n = qk + r$ where $0 \leq r < k$.

### 6.2.1 Case 1: k is even

**Lemma 18** *If $n = qk + r$ with $k$ even, then there is a Hamiltonian path in $K_p$ with $n$ edges of length $k$ and $(p-1) - n$ edges of length 1.*

**Proof:** First consider the case where $n = qk + r$ with $k$ even, $r$ even. Construct a path in the following manner:

Start at 0 and go upwards in increments of $k$ until the vertex $(q+1)k$ is reached. This picks up all of the vertices congruent to 0 modulo $k$ between 0 and $(q+1)k$. Add one to get to $(q+1)k+1$, then travel backwards in increments of $k$. This picks up all of the vertices congruent to 1 modulo $k$. At 1, add one to get to the vertex 2 and then travel to the vertices congruent to 2 modulo $k$ by way of paths of length $k$. Continue in this pattern until you have reached the vertex $r-1$. At this point, add 1 to get to the vertex $r$ and travel upwards in increments of $k$. Instead of going to $(q+1)k+r$, end at $qk+r$. Add 1 and travel downwards to $r+1$. Pick up the remaining congruence classes in this manner. Since $k$ is even, the last congruence class will be obtained by traveling downward and ending at $k-1$. This part of the Hamiltonian path containing the edges of length $k$ is summarized in the list below:

$$
\begin{aligned}
&0, k, 2k, \ldots, (q+1)k,\\
&(q+1)k+1, qk+1, \ldots, 1,\\
&2, k+2, 2k+2, \ldots, (q+1)k+2,\\
&(q+1)k+3, qk+3, \ldots, 3,\\
&\ddots\\
&r-2, k+(r-2), \ldots, (q+1)k+(r-2),\\
&(q+1)k+(r-1), qk+(r-1), \ldots, r-1,\\
&r, k+r, 2k+r, \ldots, qk+r,\\
&qk+r+1, \ldots, r+1,\\
&r+2, \ldots, qk+(r+2),\\
&\ddots\\
&qk+(k-1), \ldots, k-1
\end{aligned}
$$

One thing to note is that if $r = 0$, it is necessary to start at the row beginning with $r$ and not the row listed above as beginning with 0.

It is necessary to check that this path exhibits certain properties. First, every vertex between 0 and $(q+1)k+(r-1)$ has been visited. Furthermore, the degree of every vertex in that range is 2 with the exceptions of 0 and $k-1$. Second, the path contains exactly $n$ edges of length $k$. This can be seen since the first $r$ congruence classes have $q+1$ edges of length $k$ going forward and the last $k-r$ congruence classes contain exactly $q$ edges going forward. This gives a total of $r(q+1)+(k-r)q = qk+r = n$ edges of length $k$. Lastly, the vertex 0 has an open end, so the additional edges of length 1 can be added by subtracting 1 until we reach $(q+1)k+r$.

In the case where $k$ is even and $r$ is odd, the same construction above can be used with one slight modification. Let $r = s+1$ and use $s$ in the construction. The path contains $qk+r-1$ edges of length $k$ and ends at $k-1$. Add the edge $(k-1, -1)$. This time the open vertex is at $-1$, so the extra edges of length 1 will be added by subtraction starting at $-1$. This path contains exactly $qk+r$ edges of length $k$.

The next thing that it is necessary to show is that this construction works for all $p$. In the case of $r$ even, the largest vertex visited is $(q+1)k+r-1$. In the case of $r$

being odd, one more vertex was used, so if the vertices are relabeled so the open end is at 0, the largest vertex visited is $(q+1)k+r$. If it is the case that $(q+1)k+r \leq p-1$, then the construction will fit into $K_p$. Recall that by Lemma 17, $n = qk + r \leq \frac{p-1}{2}$ and $k \leq \frac{p-1}{2}$. It follows that, $(q+1)k + r = (qk + r) + k \leq \frac{p-1}{2} + \frac{p-1}{2} = p - 1$. Therefore, this construction works for all $p$.

■

Figures 1 and 2 show an example of the construction when $k$ is even. Figure 1 shows a path where $k$ is even and $r$ is even. In this example, 10 edges of length 4 are placed in $Z_{23}$. In this case, $n = qk + r = 2 * 4 + 2$. Figure 2 shows how the case where $k$ is even and $r$ is even can be extended to $r$ being odd by adding the edge $(k-1, -1)$. For this case we demonstrate 11 edges of length 4 in $Z_{23}$.

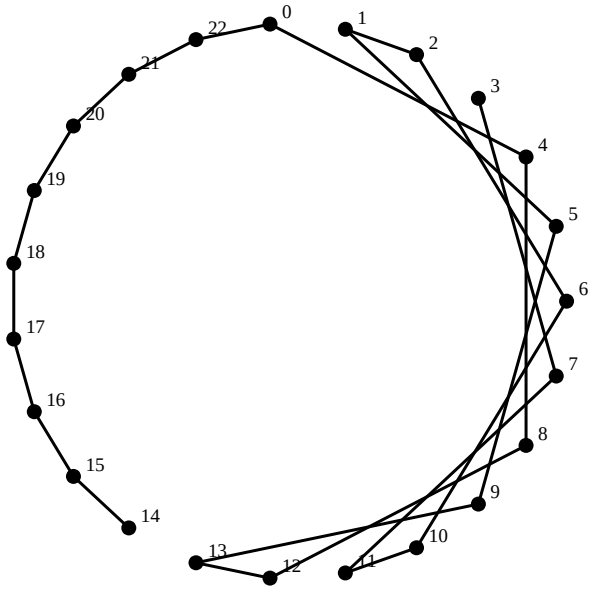


Figure 1: A Hamiltonian path with $p = 23$, $n = 10$, $k = 4$

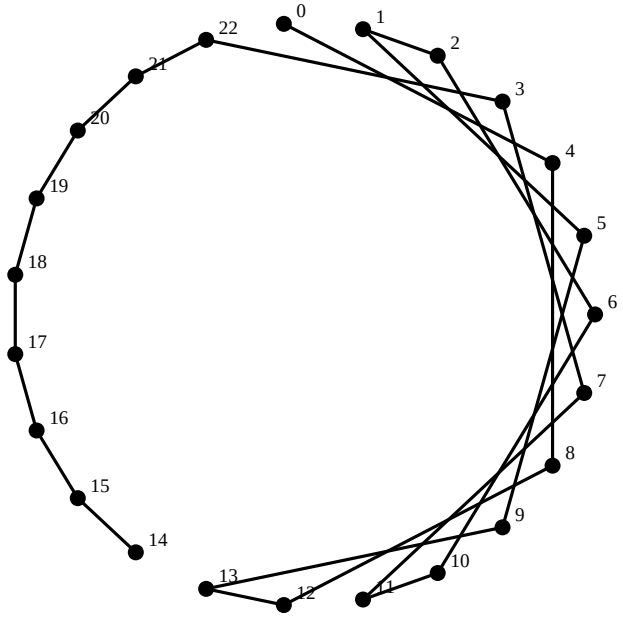


Figure 2: A Hamiltonian path with $p = 23$, $n = 11$, $k = 4$

### 6.2.2 Case 2: k is odd

It is necessary to break this case into three parts. Lemma 19 covers the case when $k$ is odd and $r$ is even. Lemmas 20 and 21 cover the case when $k$ is odd and $r$ is odd where Lemma 20 provides a construction for when $q \geq 2$ and Lemma 21 provides a construction for when $q = 1$.

**Lemma 19** *If $n = qk + r$ with $k$ odd and $r$ even, then there is a Hamiltonian path in $K_p$ with $n$ edges of length $k$ and $(p-1) - n$ edges of length 1.*

**Proof:** In the case of $k$ odd and $r$ even, we use the same pattern that was used in the $k$ even case with even remainder. One thing to note is that instead of ending the path at the vertex $k - 1$, the path ends at $qk + r - 1$. This change is noted in the summary of the pattern below.

$$
\begin{aligned}
&0, k, 2k, \ldots, (q+1)k,\\
&(q+1)k+1, qk+1, \ldots, 1,\\
&2, k+2, 2k+2, \ldots, (q+1)k+2,\\
&(q+1)k+3, qk+3, \ldots, 3,\\
&\ddots\\
&r-2, k+(r-2), \ldots, (q+1)k+(r-2),\\
&(q+1)k+(r-1), qk+(r-1), \ldots, r-1,\\
&r, k+r, 2k+r, \ldots, qk+r,\\
&qk+r+1, \ldots, r+1,\\
&r+2, \ldots, qk+(r+2),\\
&\ddots\\
&k-1, \ldots, qk+(k-1)
\end{aligned}
$$

Once again, this pattern visits the first $(q+1)k+(r-1)$ vertices and contains exactly $qk+r$ edges of length $k$ since the first $r$ congruence classes have $q+1$ edges of length $k$ going forward and the last $k-r$ congruence classes contain exactly $q$ edges going forward. The remaining edges of length 1 will be added by subtracting 1 starting at the open end, i.e. at the vertex 0.

Finally, it is necessary to check that this construction will fit into $K_p$. Once again, the highest vertex visited is $(q+1)k+r-1$. The conditions that $n = qk+r \leq \frac{p-1}{2}$ and $k \leq \frac{p-1}{2}$ still hold by Lemma 17. It follows that $(q+1)k+r-1 = (qk+r)+k-1 \leq \frac{p-1}{2} + \frac{p-1}{2} - 1 = p-2$. Therefore, this construction works for all $p$.

■

Figure 3 shows the construction in Lemma 19 for $p = 29$, $k = 5$, and $n = 12 = 2 * 5 + 2$.

**Lemma 20** *If $n = qk + r$ with $k$ odd, $r$ odd and $q \geq 2$, then a Hamiltonian path in $K_p$ with $n$ edges of length $k$ and $(p-1) - n$ edges of length 1 exists.*

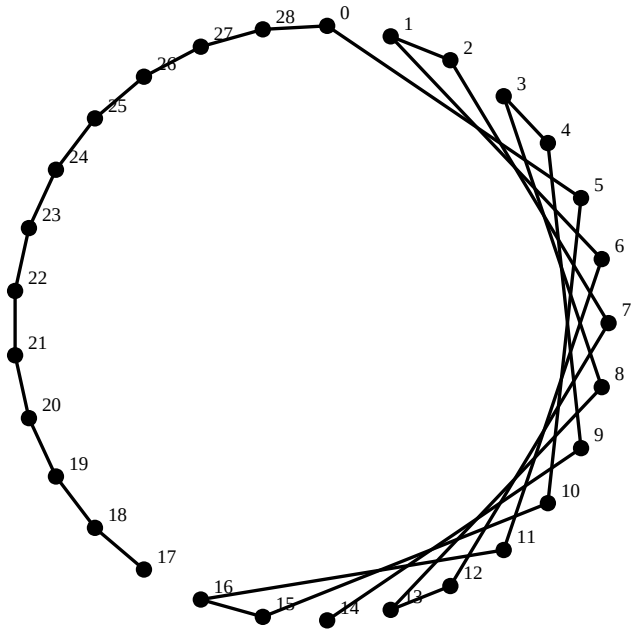


Figure 3: A Hamiltonian path with $p = 29$, $n = 12$, $k = 5$

**Proof:**

Similar to what was done with the case where $k$ is even, the construction for $k$ odd and $r$ even can be extended to $k$ odd and $r$ odd by adding one edge. As before, we do the construction for $k$ odd and $r$ even using $s = r - 1$ as the remainder. The path ends at the vertex $qk + k - 1$. Add in the edge $(qk + k - 1, qk + 2k - 1)$. The vertices between $(q+1)k+r-1$ and $(q+2)k-1$ can be reached by paths of length 1 by starting at the open end of the path at $qk+2k-1$ and subtracting 1. The vertices between $qk+2k-1$ and 0 can be reached by paths of length 1 where 0 is the starting point and 1 is subtracted.

In this case, it is necessary to show that $qk + 2k - 1 \leq p - 1$. By Lemma 17, $qk+r \leq \frac{p-1}{2}$. Since $qk \leq qk+r$, it follows that $qk \leq \frac{p-1}{2}$. It was assumed that $q \geq 2$. Therefore, $2k \leq qk$. This gives the following relation, $qk + 2k - 1 \leq qk + qk - 1 \leq \frac{p-1}{2} + \frac{p-1}{2} - 1 \leq p - 1$, which proves that this construction works for all $p$.

■

Figure 4 shows the construction from Lemma 20 for $p = 29$, $n = 13$, and $k = 5$. Note that the constructed path is the same as in Figure 3 with the addition of the edge $(14, 19)$ and the subtraction of the edge $(19, 20)$.

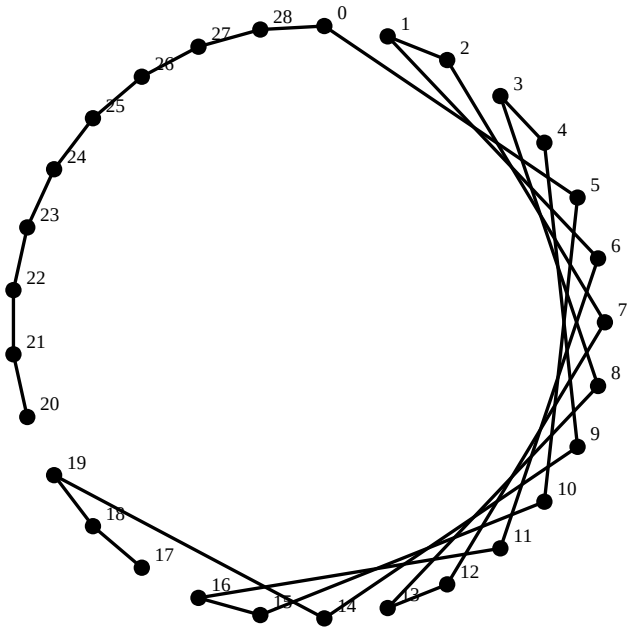


Figure 4: A Hamiltonian path with $p = 29$, $n = 13$, $k = 5$

Our final lemma concerns the case when $q = 1$.

**Lemma 21** *If $n = qk + r$ with $k$ odd, $r$ odd and $q = 1$, then a Hamiltonian path in $K_p$ with $n$ edges of length $k$ and $(p-1) - n$ edges of length 1 exists.*

**Proof:** In the case where $q = 1$, the construction in Lemma 20 will not work because in the instances when $k$ is close to $\frac{p-1}{2}$, we have $2k$ is greater than $\frac{p-1}{2}$. Consider the following construction: Start at 0 and go forward $k$. Instead of going forward 1 (as in the previous constructions), subtract 1 to get to the vertex $k - 1$. At this point

go forward $k$ to the vertex $2k-1$. Add in two edges of length 1 by going forwards to $2k$ and $2k+1$. From here, the pattern is similar to previous constructions. Travel backwards and pick up the class of numbers congruent to 1 modulo $k$. Add 1 and travel forwards by lengths of $k$, which adds the vertices congruent to 2 modulo $k$ to the path. Continue in this pattern until the vertex $r$ is reached by traveling backwards from $2k+r$. This gives $2k+r$ as the largest vertex visited thus far. At this point use the following pattern: add 1, add $k$, add 1, subtract $k$. This continues until the vertex $k-2$ is reached by traveling backwards from $2k-2$. This path is summarized in the following list:

$$
\begin{array}{l}
0, k \\
k-1, 2k-1, \\
2k, \\
2k+1, k+1, 1, \\
2, k+2, 2k+2, \\
\ddots \\
r-1, k+r-1, 2k+r-1, \\
2k+r, k+r, r, \\
r+1, k+r+1, \\
k+r+2, r+2, \\
\ddots \\
r-3, k+(r-3)k+(r-2), r-2
\end{array}
$$

This path indeed contains the correct number of edges of length $k$ since every vertex between 0 and $k+r$ has an edge of length $k$ traveling forward with the exception of vertex $k$. Also, it is easy to see that there are no isolated vertices between 0 and $k+r$. Lastly, the end at 0 is open, which enables us to pick up the vertices between $2k+r$ and 0 with edges of length 1.

The last thing to check is that the construction fits in $K_p$. By Lemma 17, $k+r \leq \frac{p-1}{2}$ and also it was assumed that $k \leq \frac{p-1}{2}$. Therefore, $2k+r = k+k+r \leq \frac{p-1}{2} + \frac{p-1}{2} = p-1$, which gives that this construction works for all $p$. ■

Figure 5 shows the construction from Lemma 21 for $p = 23$, $k = 7$, and $n = 10 = 1 * 7 + 3$.

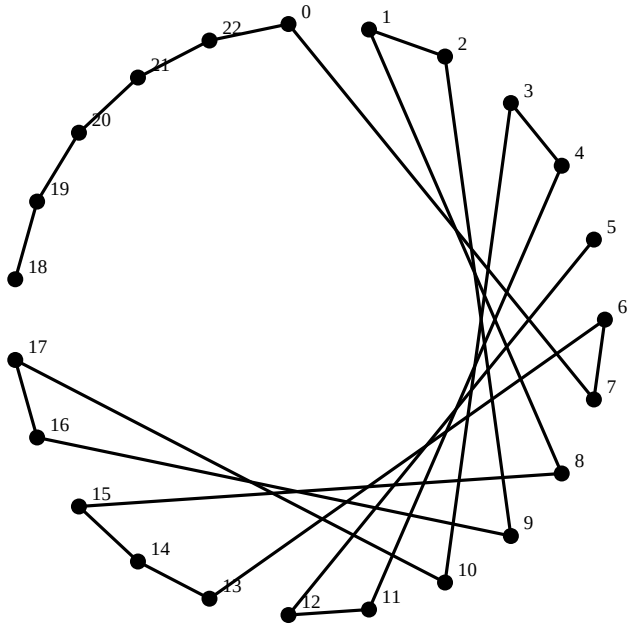


Figure 5: A Hamiltonian path with $p = 23$, $n = 10$, $k = 7$

**Theorem 22** *Given $p$ a prime, $k \leq \frac{p-1}{2}$, and $n \leq \frac{p-1}{2}$, a Hamiltonian path in $K_p$ containing $n$ edges of length $k$ and $(p-1)-n$ edges of length 1 can be constructed.*

**Proof:** This theorem follows directly from Lemmas 18, 19, 20, and 21. ■

The following Corollary is our main result.

**Corollary 23** *Let $p$ be a prime. Then for any $0 \leq n \leq p-1$ there exists a Hamiltonian path in $K_p$ containing $n$ edges of length $a$ and $(p-1)-n$ edges of length $b$ for any $1 \leq a, b \leq p-1$.*

**Proof:** By Theorem 22, there exists a Hamiltonian path in $K_p$ containing $n$ edges of length $k$ and $(p-1)-n$ edges of length 1. The existence of this path paired with Lemma 17 give the existence of the desired path. ■

We extend the result slightly in the case where the number of vertices is not a prime.

**Corollary 24** *Given $s \in \mathbb{N}$ and $n \leq \frac{s-1}{2}$, a Hamiltonian path in $K_s$ containing $n$ edges of length $k$ and $(s-1)-n$ edges of length 1 can be constructed.*

**Proof:** Examining the proofs of Lemmas 18, 19, 20, and 21, it can be observed that the condition of $s$ being a prime was not used. Therefore, the constructions work for any $s \in \mathbb{N}$. ■

**Corollary 25** *Given $s \in \mathbb{N}$ and $0 \leq n \leq s-1$, a Hamiltonian path in $K_s$ containing $n$ edges of length $a$ and $(s-1)-n$ edges of length $b$ can be constructed for any values of $a$ and $b$ such that $GCD(s,a) = 1$ and $GCD(s,b) = 1$.*

**Proof:** By Corollary 24, there exists a Hamiltonian path in $K_s$ containing $n$ edges of length $k$ and $(s-1)-n$ edges of length 1. The key ingredient used in the proof of Lemma 17 was the fact that $Z_p$ is a field when $p$ is a prime, which gives the existence of inverses. Since $a$ and $b$ are relatively prime to $s$, $a^{-1}$ and $b^{-1}$ exist in $Z_s$. Therefore, proving that the existence of a path consisting of edges of lengths $a$ and $b$ when $a$ and $b$ are relatively prime to $s$ is equivalent to the existence of a path consisting of edges of lengths $k$ and 1 is the same as the proof of Lemma 17. ■

# 7 Further Research

While the statement of Buratti's problem makes it seem relatively simple, it turns out that it is actually a very difficult problem. With the constructions presented in this paper, only a small number of the possible sets $S$ have been shown to have realizable paths. A basic computer search conducted using *Mathematica* has shown that this conjecture is true for $p = 7$. A list of one possible path for each set $S$ can be found in Appendix E along with the code used to verify that a path for each set $S$ does exist.

The problem becomes very large quickly which makes computer searches for larger values of $p$ much more difficult. The natural place to continue research into this problem is to address the case where the edges are all of three lengths.

Another area to extend research into this problem is to address cases in $K_s$ where $s$ is not a prime. Appendices F and G contain the code that gives all sets $S$ that have realizable Hamiltonian paths for $s = 8$ and $s = 9$ and the list of these sets $S$. Obviously, since neither 8 nor 9 is prime, not every multi-set $S$ results in a realizable Hamiltonian path. In the case of $s = 8$, there are 120 possible multi-sets $S$ and 105 of them are realizable as Hamiltonian paths. In the case of $s = 9$, there are 165 multi-sets $S$ with 161 of these having realizable Hamiltonian paths. The four sets that do not have realizable paths are $\{1, 3, 3, 3, 3, 3, 3, 3\}$, $\{2, 3, 3, 3, 3, 3, 3, 3\}$, $\{3, 3, 3, 3, 3, 3, 3, 3\}$, $\{3, 3, 3, 3, 3, 3, 3,$ It is easily observed that one path has all the edges of length 3 and each of these other paths have 7 edges of length 3 and one edge that is another length. These paths are not realizable because it is not possible to attain elements of more than 2 of the congruence classes modulo 3. One area of further research is to characterize which paths in $Z_n$ are realizable as Hamiltonian paths when $n$ is not a prime. We end with the following conjecture.

**Conjecture 26** *Given $p^2$ where $p$ is a prime and a multiset $S$ containing $p^2 - 1$ non-*

*zero elements from $Z_{p^2}$, there exists a Hamiltonian path in $K_{p^2}$ where the multiset of edge lengths is $S$ if $S$ contains at least $p-1$ elements that are relatively prime to $p$.*

# References


[1] A. Cayley, *On the triadic arrangements of seven and fifteen things,* London, Edinburgh, and Dublin Philosophy Magazine and Journal of Science 27 (1850) 50-53.

[2] C.J. Colbourn and J.H. Dinitz (ed.), *Handbook of Combinatorial Designs,* Chapman and Hall/CRC, Boca Raton, Florida 2007.

[3] R. Craigen and H. Kharaghani, *Hadamard Matrices and Hadamard Designs* in [2], 2007, 273-280.

[4] J.H. Dinitz, *Room Squares* in [2], 2007, 584-590.

[5] J.H. Dinitz, *Starters* in [2], 2007, 622-628.

[6] J.H. Dinitz and D.R. Stinson ed., *Contemporary Design Theory: A Collection of Surveys*, Wiley, 1992.

[7] J.H. Dinitz and D.R. Stinson, *Room Squares and Related Designs* in [6], 1992, 137-204.

[8] Y.J Ionin and T. van Trung, *Symmetric Designs* in [2], 2007, 110-124.

[9] G.B. Khosrovshani and R. Laue, *t-Designs with* $t \geq 3$, in [2], 2007, 79-101.

[10] E.S. Kramer, S.S. Magliveras, and D.R. Stinson, *Some Small Large Sets of t-Designs*, Australasian Journal of Combinatorics 3 (1991) 191-205.

[11] D.L. Kreher and D.R. Stinson, *Combinatorial Algorithms,* CRC Press, Boca Raton, Florida 1999.

[12] R. Mathon and A. Rosa, $2-(v,k,\lambda)$ *Designs of Small Order*, in [2], 2007, 25-58.

[13] D.R. Stinson, *Combinatorial Designs,* Springer, New York, 2004.

[14] T. Johnson, *Block Design for Piano,* Editions 75, Paris, 2005.

[15] T. Johnson, *Kirkman's Ladies: Rational Harmonies in Three Voices*, Editions 75, Paris, 2005.

[16] T. Johnson, *Septet,* Editions 75, Paris, 2007.

[17] W.D. Wallis, A.P. Street, and J.S. Wallis, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices,* Springer, Berlin, 1972.

[18] D. West, *Introduction to Graph Theory,* Prentice Hall, New Jersey (1996).

# A *Mathematica* Code: Characteristics of the Classes of Planes

The following initializes the combinatorica package.

```
In[31]:= << Combinatorica‘
```

We then list all 30 planes as a set of seven blocks. Each of the numbers one through 35 refers to a specific block.

```
In[32]:= squares = {{1, 8, 15, 19, 26, 28, 31},
   {1, 8, 16, 18, 25, 29, 31},
   {1, 9, 14, 19, 26, 27, 32},
   {1, 9, 16, 17, 24, 29, 32},
   {1, 10, 14, 18, 25, 27, 33},
   {1, 10, 15, 17, 24, 28, 33},
   {2, 6, 13, 20, 26, 28, 31},
   {2, 6, 16, 18, 23, 30, 31},
   {2, 7, 12, 20, 26, 27, 32},
   {2, 7, 16, 17, 22, 30, 32},
   {2, 10, 12, 18, 23, 27, 34},
   {2, 10, 13, 17, 22, 28, 34},
   {3, 5, 13, 20, 25, 29, 31},
   {3, 5, 15, 19, 23, 30, 31},
   {3, 7, 11, 20, 25, 27, 33},
   {3, 7, 15, 17, 21, 30, 33},
   {3, 9, 11, 19, 23, 27, 34},
   {3, 9, 13, 17, 21, 29, 34},
   {4, 5, 12, 20, 24, 29, 32},
   {4, 5, 14, 19, 22, 30, 32},
   {4, 6, 11, 20, 24, 28, 33},
   {4, 6, 14, 18, 21, 30, 33},
   {4, 8, 11, 19, 22, 28, 34},
   {4, 8, 12, 18, 21, 29, 34},
   {5, 10, 12, 15, 23, 24, 35},
   {5, 10, 13, 14, 22, 25, 35},
   {6, 9, 11, 16, 23, 24, 35},
   {6, 9, 13, 14, 21, 26, 35},
   {7, 8, 11, 16, 22, 25, 35},
```

```
   {7, 8, 12, 15, 21, 26, 35}};

Each of the planes above is given a number from 1-30.  Below we define a simple For loop
 that checks each pair of planes above.  The lists "disjoint", "singles", and "triples"
 keep track of which pairs of planes are disjoint, have intersection of 1, and have
 intersection of 3 respectively.  Each pair is stored as an unordered pair in the
 appropriate list.

In[33]:= disjoint = {};
singles = {};
triples = {};
For[i = 1, i <= Length[squares], i++,
 For[j = i + 1, j <= Length[squares], j++,
  {
   If[Length[Intersection[Part[squares, i], Part[squares, j]]] == 0,
    AppendTo[disjoint, {i, j}]],
   If[Length[Intersection[Part[squares, i], Part[squares, j]]] == 1,
    AppendTo[singles, {i, j}]],
   If[Length[Intersection[Part[squares, i], Part[squares, j]]] == 3,
    AppendTo[triples, {i, j}]]
   }
  ]
 ]
Length[disjoint]
disjoint
Length[singles]
singles
Length[triples]
triples

Out[37]= 120

Out[38]= {{1, 10}, {1, 11}, {1, 15}, {1, 18}, {1, 19}, {1, 22}, {1, 26}, {1, 27}, {2,
  9}, {2, 12}, {2, 16}, {2, 17}, {2, 20}, {2, 21}, {2, 25}, {2, 28}, {3,
  8}, {3, 12}, {3, 13}, {3, 16}, {3, 21}, {3, 24}, {3, 25}, {3, 29}, {4,
  7}, {4, 11}, {4, 14}, {4, 15}, {4, 22}, {4, 23}, {4, 26}, {4, 30}, {5,
  7}, {5, 10}, {5, 14}, {5, 18}, {5, 19}, {5, 23}, {5, 27}, {5, 30}, {6,
  8}, {6, 9}, {6, 13}, {6, 17}, {6, 20}, {6, 24}, {6, 28}, {6, 29}, {7,
  16}, {7, 17}, {7, 20}, {7, 24}, {7, 25}, {7, 29}, {8, 15}, {8, 18}, {8,
  19}, {8, 23}, {8, 26}, {8, 30}, {9, 14}, {9, 18}, {9, 22}, {9, 23}, {9,
```

```
  26}, {9, 27}, {10, 13}, {10, 17}, {10, 21}, {10, 24}, {10, 25}, {10,
  28}, {11, 13}, {11, 16}, {11, 20}, {11, 21}, {11, 28}, {11, 29}, {12,
  14}, {12, 15}, {12, 19}, {12, 22}, {12, 27}, {12, 30}, {13, 22}, {13,
  23}, {13, 27}, {13, 30}, {14, 21}, {14, 24}, {14, 28}, {14, 29}, {15,
  20}, {15, 24}, {15, 25}, {15, 28}, {16, 19}, {16, 23}, {16, 26}, {16,
  27}, {17, 19}, {17, 22}, {17, 26}, {17, 30}, {18, 20}, {18, 21}, {18,
  25}, {18, 29}, {19, 28}, {19, 29}, {20, 27}, {20, 30}, {21, 26}, {21,
  30}, {22, 25}, {22, 29}, {23, 25}, {23, 28}, {24, 26}, {24, 27}}

Out[39]= 210

Out[40]= {{1, 4}, {1, 5}, {1, 8}, {1, 9}, {1, 12}, {1, 13}, {1, 16}, {1, 17}, {1,
  20}, {1, 21}, {1, 24}, {1, 25}, {1, 28}, {1, 29}, {2, 3}, {2, 6}, {2,
  7}, {2, 10}, {2, 11}, {2, 14}, {2, 15}, {2, 18}, {2, 19}, {2, 22}, {2,
  23}, {2, 26}, {2, 27}, {2, 30}, {3, 6}, {3, 7}, {3, 10}, {3, 11}, {3,
  14}, {3, 15}, {3, 18}, {3, 19}, {3, 22}, {3, 23}, {3, 26}, {3, 27}, {3,
  30}, {4, 5}, {4, 8}, {4, 9}, {4, 12}, {4, 13}, {4, 16}, {4, 17}, {4,
  20}, {4, 21}, {4, 24}, {4, 25}, {4, 28}, {4, 29}, {5, 8}, {5, 9}, {5,
  12}, {5, 13}, {5, 16}, {5, 17}, {5, 20}, {5, 21}, {5, 24}, {5, 25}, {5,
  28}, {5, 29}, {6, 7}, {6, 10}, {6, 11}, {6, 14}, {6, 15}, {6, 18}, {6,
  19}, {6, 22}, {6, 23}, {6, 26}, {6, 27}, {6, 30}, {7, 10}, {7, 11}, {7,
  14}, {7, 15}, {7, 18}, {7, 19}, {7, 22}, {7, 23}, {7, 26}, {7, 27}, {7,
  30}, {8, 9}, {8, 12}, {8, 13}, {8, 16}, {8, 17}, {8, 20}, {8, 21}, {8,
  24}, {8, 25}, {8, 28}, {8, 29}, {9, 12}, {9, 13}, {9, 16}, {9, 17}, {9,
  20}, {9, 21}, {9, 24}, {9, 25}, {9, 28}, {9, 29}, {10, 11}, {10, 14}, {10,
  15}, {10, 18}, {10, 19}, {10, 22}, {10, 23}, {10, 26}, {10, 27}, {10,
  30}, {11, 14}, {11, 15}, {11, 18}, {11, 19}, {11, 22}, {11, 23}, {11,
  26}, {11, 27}, {11, 30}, {12, 13}, {12, 16}, {12, 17}, {12, 20}, {12,
  21}, {12, 24}, {12, 25}, {12, 28}, {12, 29}, {13, 16}, {13, 17}, {13,
  20}, {13, 21}, {13, 24}, {13, 25}, {13, 28}, {13, 29}, {14, 15}, {14,
  18}, {14, 19}, {14, 22}, {14, 23}, {14, 26}, {14, 27}, {14, 30}, {15,
  18}, {15, 19}, {15, 22}, {15, 23}, {15, 26}, {15, 27}, {15, 30}, {16,
  17}, {16, 20}, {16, 21}, {16, 24}, {16, 25}, {16, 28}, {16, 29}, {17,
  20}, {17, 21}, {17, 24}, {17, 25}, {17, 28}, {17, 29}, {18, 19}, {18,
  22}, {18, 23}, {18, 26}, {18, 27}, {18, 30}, {19, 22}, {19, 23}, {19,
  26}, {19, 27}, {19, 30}, {20, 21}, {20, 24}, {20, 25}, {20, 28}, {20,
  29}, {21, 24}, {21, 25}, {21, 28}, {21, 29}, {22, 23}, {22, 26}, {22,
  27}, {22, 30}, {23, 26}, {23, 27}, {23, 30}, {24, 25}, {24, 28}, {24,
  29}, {25, 28}, {25, 29}, {26, 27}, {26, 30}, {27, 30}, {28, 29}}
```

```
Out[41]= 105

Out[42]= {{1, 2}, {1, 3}, {1, 6}, {1, 7}, {1, 14}, {1, 23}, {1, 30}, {2, 4}, {2,
  5}, {2, 8}, {2, 13}, {2, 24}, {2, 29}, {3, 4}, {3, 5}, {3, 9}, {3, 17}, {3,
  20}, {3, 28}, {4, 6}, {4, 10}, {4, 18}, {4, 19}, {4, 27}, {5, 6}, {5,
  11}, {5, 15}, {5, 22}, {5, 26}, {6, 12}, {6, 16}, {6, 21}, {6, 25}, {7,
  8}, {7, 9}, {7, 12}, {7, 13}, {7, 21}, {7, 28}, {8, 10}, {8, 11}, {8,
  14}, {8, 22}, {8, 27}, {9, 10}, {9, 11}, {9, 15}, {9, 19}, {9, 30}, {10,
  12}, {10, 16}, {10, 20}, {10, 29}, {11, 12}, {11, 17}, {11, 24}, {11,
  25}, {12, 18}, {12, 23}, {12, 26}, {13, 14}, {13, 15}, {13, 18}, {13,
  19}, {13, 26}, {14, 16}, {14, 17}, {14, 20}, {14, 25}, {15, 16}, {15,
  17}, {15, 21}, {15, 29}, {16, 18}, {16, 22}, {16, 30}, {17, 18}, {17,
  23}, {17, 27}, {18, 24}, {18, 28}, {19, 20}, {19, 21}, {19, 24}, {19,
  25}, {20, 22}, {20, 23}, {20, 26}, {21, 22}, {21, 23}, {21, 27}, {22,
  24}, {22, 28}, {23, 24}, {23, 29}, {24, 30}, {25, 26}, {25, 27}, {25,
  30}, {26, 28}, {26, 29}, {27, 28}, {27, 29}, {28, 30}, {29, 30}}
```

The vertices of the graph below correspond to the 30 different planes. An edge is drawn between any pair that has intersection of exactly 1. The command "MaximumClique" found that the largest clique in the graph has fifteen vertices and returned one such clique. The command "CliqueQ" then verified that the fifteen planes not in the first clique returned also formed a clique. Lastly, the degree sequence shows that each vertex has degreee 14, which proves that there are no edges between the cliques.

```
In[44]:= g = FromUnorderedPairs[singles];
ShowGraph[g]
MaximumClique[g]
CliqueQ[g, {2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30}]
Degrees[g]

Out[45]= {1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29}

Out[46]= True

Out[47]= {14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, \
14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14}
```

The following code defines a graph with the vertices corresponding to the 30 planes and

```
 edges being drawn between two vertices if the corresponding planes are disjoint from
  each other.  The maximum clique in this graph is of size 2 and each vertex has degree
   8.  This proves that the largest set of planes with all distinct blocks is 2 and given
    any plane there are exactly 8 planes from which it is disjoint.

In[48]:= edges = disjoint;
h = FromUnorderedPairs[edges];
ShowGraph[h]
MaximumClique[h]
Degrees[h]

Out[50]= {1, 10}

Out[51]= {8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, \
8, 8, 8, 8, 8}

This last graph is defined with the vertices being the 30 planes and the edges connecting
 two planes who have intersection of exactly 3.  Every vertex having degree 7 proves that
  given any plane, it has intersection of 3 with exactly 7 planes.

In[52]:= edges = triples;
k = FromUnorderedPairs[edges];
ShowGraph[k]
MaximumClique[k]
Degrees[k]


Out[54]= {1, 2}

Out[55]= {7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, \
7, 7, 7, 7, 7}
```

# B *Mathematica* Code: Sets of 2,3,4,5, and 6 Planes with the Maximum Number of Distinct Blocks

We first initialize the Combinatorica package and define the 30 Fano planes as sets of 7 blocks.

```
<< Combinatorica‘

squares = {{1, 8, 15, 19, 26, 28, 31},
   {1, 8, 16, 18, 25, 29, 31},
   {1, 9, 14, 19, 26, 27, 32},
   {1, 9, 16, 17, 24, 29, 32},
   {1, 10, 14, 18, 25, 27, 33},
   {1, 10, 15, 17, 24, 28, 33},
   {2, 6, 13, 20, 26, 28, 31},
   {2, 6, 16, 18, 23, 30, 31},
   {2, 7, 12, 20, 26, 27, 32},
   {2, 7, 16, 17, 22, 30, 32},
   {2, 10, 12, 18, 23, 27, 34},
   {2, 10, 13, 17, 22, 28, 34},
   {3, 5, 13, 20, 25, 29, 31},
   {3, 5, 15, 19, 23, 30, 31},
   {3, 7, 11, 20, 25, 27, 33},
   {3, 7, 15, 17, 21, 30, 33},
   {3, 9, 11, 19, 23, 27, 34},
   {3, 9, 13, 17, 21, 29, 34},
   {4, 5, 12, 20, 24, 29, 32},
   {4, 5, 14, 19, 22, 30, 32},
   {4, 6, 11, 20, 24, 28, 33},
   {4, 6, 14, 18, 21, 30, 33},
   {4, 8, 11, 19, 22, 28, 34},
   {4, 8, 12, 18, 21, 29, 34},
   {5, 10, 12, 15, 23, 24, 35},
   {5, 10, 13, 14, 22, 25, 35},
   {6, 9, 11, 16, 23, 24, 35},
   {6, 9, 13, 14, 21, 26, 35},
   {7, 8, 11, 16, 22, 25, 35},
   {7, 8, 12, 15, 21, 26, 35}};
```

```
{{1, 8, 15, 19, 26, 28, 31}, {1, 8, 16, 18, 25, 29, 31}, {1, 9, 14, 19, 26,
  27, 32}, {1, 9, 16, 17, 24, 29, 32}, {1, 10, 14, 18, 25, 27, 33}, {1, 10,
  15, 17, 24, 28, 33}, {2, 6, 13, 20, 26, 28, 31}, {2, 6, 16, 18, 23, 30,
  31}, {2, 7, 12, 20, 26, 27, 32}, {2, 7, 16, 17, 22, 30, 32}, {2, 10, 12, 18,
   23, 27, 34}, {2, 10, 13, 17, 22, 28, 34}, {3, 5, 13, 20, 25, 29, 31}, {3,
  5, 15, 19, 23, 30, 31}, {3, 7, 11, 20, 25, 27, 33}, {3, 7, 15, 17, 21, 30,
  33}, {3, 9, 11, 19, 23, 27, 34}, {3, 9, 13, 17, 21, 29, 34}, {4, 5, 12, 20,
  24, 29, 32}, {4, 5, 14, 19, 22, 30, 32}, {4, 6, 11, 20, 24, 28, 33}, {4, 6,
  14, 18, 21, 30, 33}, {4, 8, 11, 19, 22, 28, 34}, {4, 8, 12, 18, 21, 29,
  34}, {5, 10, 12, 15, 23, 24, 35}, {5, 10, 13, 14, 22, 25, 35}, {6, 9, 11,
  16, 23, 24, 35}, {6, 9, 13, 14, 21, 26, 35}, {7, 8, 11, 16, 22, 25, 35}, {7,
   8, 12, 15, 21, 26, 35}}
```

Each subset of 3 planes from the list above was checked for how many distinct blocks it contained. If it contained the maximum number of 20, the set was appended to a list. The length of the list was then calculated. It was then verified that there did not exist a set that contained more than 20 blocks. This process was repeated for subsets of size 4, 5, and 6.

```
k3sets = Subsets[squares, {3}];
lengths = {};
large = {};
For[i = 1, i <= Length[k3sets], i++,
 {threes = Part[k3sets, i],
  union = Union[Part[threes, 1], Part[threes, 2], Part[threes, 3]],
  If[Length[union] == 20, AppendTo[lengths, i]]}
 ]
For[i = 1, i <= Length[k3sets], i++,
 {threes = Part[k3sets, i],
  union = Union[Part[threes, 1], Part[threes, 2], Part[threes, 3]],
  If[Length[union] > 20, AppendTo[large, i]]}
 ]
Length[lengths]
Length[large]
```

```
840
```

```
0
```

```
k4sets = Subsets[squares, {4}];
parts = {};
large = {};
For[i = 1, i <= Length[k4sets], i++,
 {fours = Part[k4sets, i],
  union = Union[Part[fours, 1], Part[fours, 2], Part[fours, 3],
    Part[fours, 4]], If[Length[union] == 26, AppendTo[parts, i]]}
 ]
For[i = 1, i <= Length[k4sets], i++,
 {fours = Part[k4sets, i],
  union = Union[Part[fours, 1], Part[fours, 2], Part[fours, 3],
    Part[fours, 4]], If[Length[union] > 26, AppendTo[large, i]]}
 ]
Length[parts]
Length[large]
```

```
k5sets = Subsets[squares, {5}];
pieces = {};
large = {};
For[i = 1, i <= Length[k5sets], i++,
 {fives = Part[k5sets, i],
  union = Union[Part[fives, 1], Part[fives, 2], Part[fives, 3],
    Part[fives, 4], Part[fives, 5]],
  If[Length[union] == 31, AppendTo[pieces, i]]}
 ]
For[i = 1, i <= Length[k5sets], i++,
 {fives = Part[k5sets, i],
  union = Union[Part[fives, 1], Part[fives, 2], Part[fives, 3],
    Part[fives, 4], Part[fives, 5]],
  If[Length[union] > 31, AppendTo[large, i]]}
 ]
Length[pieces]
Length[large]
```

```
840

k6sets = Subsets[squares, {6}];
pieces6 = {};
large = {};
For[i = 1, i <= Length[k6sets], i++,
 {sixes = Part[k6sets, i],
  union = Union[Part[sixes, 1], Part[sixes, 2], Part[sixes, 3],
    Part[sixes, 4], Part[sixes, 5], Part[sixes, 6]],
  If[Length[union] == 35, AppendTo[pieces6, i]]}
 ]
Length[pieces6]

210
```

# C *Mathematica* Code: Isomorphism Classes of sets of 2,3,4,5, and 6 Planes with the Maximum Number of Distinct Blocks

Initialize the Combinatorica package and define the 30 Fano planes. The planes are divided into the two classes.

```
In[227]:= << Combinatorica`

In[228]:= B1 = {{1, 2, 7}, {1, 3, 6}, {1, 4, 5}, {2, 3, 5}, {2, 4, 6}, {3, 4, 7}, {5, 6,
     7}};
B2 = {{1, 2, 6}, {1, 3, 5}, {1, 4, 7}, {2, 3, 7}, {2, 4, 5}, {3, 4, 6}, {5, 6,
     7}};
B3 = {{1, 2, 5}, {1, 3, 7}, {1, 4, 6}, {2, 3, 6}, {2, 4, 7}, {3, 4, 5}, {5, 6,
     7}};
B4 = {{1, 2, 7}, {1, 3, 4}, {1, 5, 6}, {2, 3, 6}, {2, 4, 5}, {3, 5, 7}, {4, 6,
     7}};
B5 = {{1, 2, 6}, {1, 3, 7}, {1, 4, 5}, {2, 3, 4}, {2, 5, 7}, {3, 5, 6}, {4, 6,
     7}};
B6 = {{1, 2, 4}, {1, 3, 6}, {1, 5, 7}, {2, 3, 7}, {2, 5, 6}, {3, 4, 5}, {4, 6,
     7}};
B7 = {{1, 2, 7}, {1, 3, 5}, {1, 4, 6}, {2, 3, 4}, {2, 5, 6}, {3, 6, 7}, {4, 5,
     7}};
B8 = {{1, 2, 5}, {1, 3, 4}, {1, 6, 7}, {2, 3, 7}, {2, 4, 6}, {3, 5, 6}, {4, 5,
     7}};
B9 = {{1, 2, 4}, {1, 3, 7}, {1, 5, 6}, {2, 3, 5}, {2, 6, 7}, {3, 4, 6}, {4, 5,
     7}};
B10 = {{1, 2, 6}, {1, 3, 4}, {1, 5, 7}, {2, 3, 5}, {2, 4, 7}, {3, 6, 7}, {4,
    5, 6}};
B11 = {{1, 2, 5}, {1, 3, 6}, {1, 4, 7}, {2, 3, 4}, {2, 6, 7}, {3, 5, 7}, {4,
    5, 6}};
B12 = {{1, 2, 4}, {1, 3, 5}, {1, 6, 7}, {2, 3, 6}, {2, 5, 7}, {3, 4, 7}, {4,
    5, 6}};
B13 = {{1, 2, 3}, {1, 4, 7}, {1, 5, 6}, {2, 4, 6}, {2, 5, 7}, {3, 4, 5}, {3,
    6, 7}};
B14 = {{1, 2, 3}, {1, 4, 5}, {1, 6, 7}, {2, 4, 7}, {2, 5, 6}, {3, 4, 6}, {3,
    5, 7}};
B15 = {{1, 2, 3}, {1, 4, 6}, {1, 5, 7}, {2, 4, 5}, {2, 6, 7}, {3, 4, 7}, {3,
```

```
    5, 6}};
B = {B1, B2, B3, B4, B5, B6, B7, B8, B9, B10, B11, B12, B13, B14, B15};

In[244]:= A1 = {{1, 2, 7}, {1, 3, 5}, {1, 4, 6}, {2, 3, 6}, {2, 4, 5}, {3, 4, 7}, {5, 6,
     7}};
A2 = {{1, 2, 6}, {1, 3, 7}, {1, 4, 5}, {2, 3, 5}, {2, 4, 7}, {3, 4, 6}, {5, 6,
     7}};
A3 = {{1, 2, 5}, {1, 3, 6}, {1, 4, 7}, {2, 3, 7}, {2, 4, 6}, {3, 4, 5}, {5, 6,
     7}};
A4 = {{1, 2, 7}, {1, 3, 6}, {1, 4, 5}, {2, 3, 4}, {2, 5, 6}, {3, 5, 7}, {4, 6,
     7}};
A5 = {{1, 2, 6}, {1, 3, 4}, {1, 5, 7}, {2, 3, 7}, {2, 4, 5}, {3, 5, 6}, {4, 6,
     7}};
A6 = {{1, 2, 4}, {1, 3, 7}, {1, 5, 6}, {2, 3, 6}, {2, 5, 7}, {3, 4, 5}, {4, 6,
     7}};
A7 = {{1, 2, 7}, {1, 3, 4}, {1, 5, 6}, {2, 3, 5}, {2, 4, 6}, {3, 6, 7}, {4, 5,
     7}};
A8 = {{1, 2, 5}, {1, 3, 7}, {1, 4, 6}, {2, 3, 4}, {2, 6, 7}, {3, 5, 6}, {4, 5,
     7}};
A9 = {{1, 2, 4}, {1, 3, 5}, {1, 6, 7}, {2, 3, 7}, {2, 5, 6}, {3, 4, 6}, {4, 5,
     7}};
A10 = {{1, 2, 6}, {1, 3, 5}, {1, 4, 7}, {2, 3, 4}, {2, 5, 7}, {3, 6, 7}, {4,
    5, 6}};
A11 = {{1, 2, 5}, {1, 3, 4}, {1, 6, 7}, {2, 3, 6}, {2, 4, 7}, {3, 5, 7}, {4,
    5, 6}};
A12 = {{1, 2, 4}, {1, 3, 6}, {1, 5, 7}, {2, 3, 5}, {2, 6, 7}, {3, 4, 7}, {4,
    5, 6}};
A13 = {{1, 2, 3}, {1, 4, 6}, {1, 5, 7}, {2, 4, 7}, {2, 5, 6}, {3, 4, 5}, {3,
    6, 7}};
A14 = {{1, 2, 3}, {1, 4, 7}, {1, 5, 6}, {2, 4, 5}, {2, 6, 7}, {3, 4, 6}, {3,
    5, 7}};
A15 = {{1, 2, 3}, {1, 4, 5}, {1, 6, 7}, {2, 4, 6}, {2, 5, 7}, {3, 4, 7}, {3,
    5, 6}};
A = {A1, A2, A3, A4, A5, A6, A7, A8, A9, A10, A11, A12, A13, A14, A15};
```

```
All possible blocks are created below and then numbered.  The planes are redefined in
 terms of seven numbers, each referring to a block, instead of a list of blocks.

In[260]:= points = {1, 2, 3, 4, 5, 6, 7};
blocks = Subsets[points, {3}]
Length[blocks]

Out[261]= {{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}, {1, 2, 7}, {1, 3, 4}, {1, 3,
  5}, {1, 3, 6}, {1, 3, 7}, {1, 4, 5}, {1, 4, 6}, {1, 4, 7}, {1, 5, 6}, {1, 5,
   7}, {1, 6, 7}, {2, 3, 4}, {2, 3, 5}, {2, 3, 6}, {2, 3, 7}, {2, 4, 5}, {2,
  4, 6}, {2, 4, 7}, {2, 5, 6}, {2, 5, 7}, {2, 6, 7}, {3, 4, 5}, {3, 4, 6}, {3,
   4, 7}, {3, 5, 6}, {3, 5, 7}, {3, 6, 7}, {4, 5, 6}, {4, 5, 7}, {4, 6,
  7}, {5, 6, 7}}

Out[262]= 35

In[263]:= For[i = 1, i <= Length[blocks],
 i++, {part = Part[blocks, i], A = A /. part -> i, B = B /. part -> i}]
A;
B;
Planes = Union[A, B]
Length[Planes]

Out[266]= {{1, 10, 15, 21, 24, 28, 29}, {1, 10, 15, 22, 23, 27, 30}, {1, 11, 14, 20, 25,
   28, 29}, {1, 11, 14, 22, 23, 26, 31}, {1, 12, 13, 20, 25, 27, 30}, {1, 12,
  13, 21, 24, 26, 31}, {2, 7, 15, 18, 24, 28, 32}, {2, 7, 15, 19, 23, 27,
  33}, {2, 8, 14, 17, 25, 28, 32}, {2, 8, 14, 19, 23, 26, 34}, {2, 9, 13, 17,
  25, 27, 33}, {2, 9, 13, 18, 24, 26, 34}, {3, 6, 15, 18, 22, 30, 32}, {3, 6,
  15, 19, 21, 29, 33}, {3, 8, 12, 16, 25, 30, 32}, {3, 8, 12, 19, 21, 26,
  35}, {3, 9, 11, 16, 25, 29, 33}, {3, 9, 11, 18, 22, 26, 35}, {4, 6, 14, 17,
  22, 31, 32}, {4, 6, 14, 19, 20, 29, 34}, {4, 7, 12, 16, 24, 31, 32}, {4, 7,
  12, 19, 20, 27, 35}, {4, 9, 10, 16, 24, 29, 34}, {4, 9, 10, 17, 22, 27,
  35}, {5, 6, 13, 17, 21, 31, 33}, {5, 6, 13, 18, 20, 30, 34}, {5, 7, 11, 16,
  23, 31, 33}, {5, 7, 11, 18, 20, 28, 35}, {5, 8, 10, 16, 23, 30, 34}, {5, 8,
  10, 17, 21, 28, 35}}

Out[267]= 30

All permutations on 7 elements are listed and given a number.  They are then converted to
 the form we need to use them later.
```

```
In[268]:= allpermutations = Permutations[{1, 2, 3, 4, 5, 6, 7}, {7}];
allperms = {};
For[i = 1, i <= Length[allpermutations], i++,
 {permutation = Part[allpermutations, i],
  perm = {},
  For[j = 1, j <= Length[blocks], j++,
   {new =
     Part[blocks, j] /. {1 -> Part[permutation, 1], 2 -> Part[permutation, 2],
        3 -> Part[permutation, 3], 4 -> Part[permutation, 4],
       5 -> Part[permutation, 5], 6 -> Part[permutation, 6],
       7 -> Part[permutation, 7]},
    For[k = 1, k <= 35, k++,
     If[Sort[new] == Part[blocks, k], AppendTo[perm, j -> k]]]
    }
   ],
  AppendTo[allperms, perm]
  }
 ]
```

We let each of the permutations above act on plane i. Each of the lists "permsi" for 1<= i <= 30 keeps track of where each permutation takes the plane i.

```
In[271]:= perms1 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 1],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms1, j], i]
    ]
   ]
  }
 ]
```

```
In[273]:= perms2 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
```

```
  plane = Part[Planes, 2],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms2, j], i]
    ]
   ]
  }
 ]


In[275]:= perms3 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 3],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms3, j], i]
    ]
   ]
  }
 ]

In[277]:= perms4 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 4],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms4, j], i]
    ]
   ]
  }
 ]

In[279]:= perms5 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
```

```
  plane = Part[Planes, 5],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms5, j], i]
    ]
   ]
  }
 ]

In[281]:= perms6 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 6],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms6, j], i]
    ]
   ]
  }
 ]

In[283]:= perms7 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 7],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms7, j], i]
    ]
   ]
  }
 ]

In[285]:= perms8 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 8],
```

```
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms8, j], i]
    ]
   ]
  }
 ]

In[287]:= perms9 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 9],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms9, j], i]
    ]
   ]
  }
 ]

In[289]:= perms10 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 10],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms10, j], i]
    ]
   ]
  }
 ]

In[291]:= perms11 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 11],
  new = plane /. perm,
```

```
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms11, j], i]
    ]
   ]
  }
 ]

In[293]:= perms12 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 12],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms12, j], i]
    ]
   ]
  }
 ]

In[295]:= perms13 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 13],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms13, j], i]
    ]
   ]
  }
 ]

In[297]:= perms14 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 14],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
```

```
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms14, j], i]
    ]
   ]
  }
 ]

In[299]:= perms15 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 15],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms15, j], i]
    ]
   ]
  }
 ]

In[301]:= perms16 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 16],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms16, j], i]
    ]
   ]
  }
 ]

In[303]:= perms17 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 17],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms17, j], i]
```

```
    ]
   ]
  }
 ]

In[305]:= perms18 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 18],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms18, j], i]
    ]
   ]
  }
 ]

In[307]:= perms19 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 19],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms19, j], i]
    ]
   ]
  }
 ]

In[309]:= perms20 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 20],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms20, j], i]
    ]
```

```
   ]
  }
 ]

In[311]:= perms21 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 21],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms21, j], i]
    ]
   ]
  }
 ]

In[313]:= perms22 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 22],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms22, j], i]
    ]
   ]
  }
 ]

In[315]:= perms23 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 23],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms23, j], i]
    ]
   ]
```

```
  }
 ]

In[317]:= perms24 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 24],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms24, j], i]
    ]
   ]
  }
 ]

In[319]:= perms25 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 25],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms25, j], i]
    ]
   ]
  }
 ]

In[321]:= perms26 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 26],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms26, j], i]
    ]
   ]
  }
```

```
 ]

In[323]:= perms27 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 27],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms27, j], i]
    ]
   ]
  }
 ]

In[325]:= perms28 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 28],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms28, j], i]
    ]
   ]
  }
 ]

In[327]:= perms29 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 29],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms29, j], i]
    ]
   ]
  }
 ]
```

```
In[329]:= perms30 = {{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, \
{}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}};
For[i = 1, i <= Length[allperms], i++,
 {perm = Part[allperms, i],
  plane = Part[Planes, 30],
  new = plane /. perm,
  For[j = 1, j <= 30, j++,
   If[Sort[new] == Part[Planes, j], AppendTo[Part[perms30, j], i]
    ]
   ]
  }
 ]
```

The list "perms" keeps track of what the image of each permutation on each plane is.
Part[Part[perms,i,],j] gives the permutations that take plane i to plane j.

```
In[331]:= perms = {perms1, perms2, perms3, perms4, perms5, perms6, perms7, perms8,
   perms9, perms10, perms11, perms12, perms13, perms14, perms15, perms16,
   perms17, perms18, perms19, perms20, perms21, perms22, perms23, perms24,
   perms25, perms26, perms27, perms28, perms29, perms30};
Length[perms]
Length[Part[perms, 1]]
Length[Part[Part[perms, 1], 1]]

Out[332]= 30

Out[333]= 30

Out[334]= 168
```

A list of all pairs of disjoint planes is below. We designate the pair {1,10} as our "starter pair." We then show that there is some permutation that takes every pair {a,b} to {1,10}. This process is repeated for sets of 3, 4, 5 and 6 planes with the maximum number of distinct blocks.

```
In[335]:= disjointplanes = {};
For[i = 1, i <= 30, i++,
```

```
 For[j = i + 1, j <= 30, j++,
  If[Length[Union[Part[Planes, i], Part[Planes, j]]] == 14,
   AppendTo[disjointplanes, {i, j}]
   ]
  ]
 ]
disjointplanes
Length[disjointplanes]

Out[337]= {{1, 10}, {1, 11}, {1, 15}, {1, 18}, {1, 19}, {1, 22}, {1, 26}, {1, 27}, {2,
  9}, {2, 12}, {2, 16}, {2, 17}, {2, 20}, {2, 21}, {2, 25}, {2, 28}, {3,
  8}, {3, 12}, {3, 13}, {3, 16}, {3, 21}, {3, 24}, {3, 25}, {3, 29}, {4,
  7}, {4, 11}, {4, 14}, {4, 15}, {4, 22}, {4, 23}, {4, 26}, {4, 30}, {5,
  7}, {5, 10}, {5, 14}, {5, 18}, {5, 19}, {5, 23}, {5, 27}, {5, 30}, {6,
  8}, {6, 9}, {6, 13}, {6, 17}, {6, 20}, {6, 24}, {6, 28}, {6, 29}, {7,
  16}, {7, 17}, {7, 20}, {7, 24}, {7, 25}, {7, 29}, {8, 15}, {8, 18}, {8,
  19}, {8, 23}, {8, 26}, {8, 30}, {9, 14}, {9, 18}, {9, 22}, {9, 23}, {9,
  26}, {9, 27}, {10, 13}, {10, 17}, {10, 21}, {10, 24}, {10, 25}, {10,
  28}, {11, 13}, {11, 16}, {11, 20}, {11, 21}, {11, 28}, {11, 29}, {12,
  14}, {12, 15}, {12, 19}, {12, 22}, {12, 27}, {12, 30}, {13, 22}, {13,
  23}, {13, 27}, {13, 30}, {14, 21}, {14, 24}, {14, 28}, {14, 29}, {15,
  20}, {15, 24}, {15, 25}, {15, 28}, {16, 19}, {16, 23}, {16, 26}, {16,
  27}, {17, 19}, {17, 22}, {17, 26}, {17, 30}, {18, 20}, {18, 21}, {18,
  25}, {18, 29}, {19, 28}, {19, 29}, {20, 27}, {20, 30}, {21, 26}, {21,
  30}, {22, 25}, {22, 29}, {23, 25}, {23, 28}, {24, 26}, {24, 27}}

Out[338]= 120

In[339]:= startpair = {1, 10};
isomorphic = {};
noniso = {};
For[i = 1, i <= Length[disjointplanes], i++,
 {pair = Part[disjointplanes, i],
  If[Length[
     Intersection[Part[Part[perms, 1], Part[pair, 1]],
      Part[Part[perms, 10, Part[pair, 2]]]]] >= 1,
   AppendTo[isomorphic, pair],
   If[Length[
      Intersection[Part[Part[perms, 1], Part[pair, 2]],
       Part[Part[perms, 10, Part[pair, 1]]]]] >= 1,
```

```
    AppendTo[isomorphic, pair],
    AppendTo[noniso, pair]]
   ]
  }
 ]
Length[isomorphic]
Length[noniso]

Out[343]= 120

Out[344]= 0

.

In[345]:= k3sets = Subsets[Planes, {3}];
lengths = {};
For[i = 1, i <= Length[k3sets], i++,
 {threes = Part[k3sets, i],
  union = Union[Part[threes, 1], Part[threes, 2], Part[threes, 3]],
  If[Length[union] == 20, AppendTo[lengths, i]]}
 ]



In[348]:= sets = {};
For[i = 1, i <= Length[lengths], i++,
 AppendTo[sets, Part[k3sets, Part[lengths, i]]]]
setsasblocks = sets;
For[i = 1, i <= 30, i++,
 sets = sets /. Part[Planes, i] -> i]
sets;

In[353]:= Length[sets]

Out[353]= 840

In[354]:= start = {{1, 10, 15, 21, 24, 28, 29}, {1, 11, 14, 22, 23, 26, 31}, {2, 9, 13,
   17, 25, 27, 33}}

Out[354]= {{1, 10, 15, 21, 24, 28, 29}, {1, 11, 14, 22, 23, 26, 31}, {2, 9, 13, 17, 25,
```

```
  27, 33}}

In[355]:= a1 = Part[start, 1]
a2 = Part[start, 2]
b = Part[start, 3]

Out[355]= {1, 10, 15, 21, 24, 28, 29}

Out[356]= {1, 11, 14, 22, 23, 26, 31}

Out[357]= {2, 9, 13, 17, 25, 27, 33}

In[358]:= part = Part[k3sets, Part[lengths, 2]]


Out[358]= {{1, 10, 15, 21, 24, 28, 29}, {1, 11, 14, 22, 23, 26, 31}, {3, 8, 12, 16, 25,
  30, 32}}

In[359]:=
If[Length[Union[Part[part, 1], Part[part, 2]]] == 13, {isob = Part[part, 3],
  isoa1 = Part[part, 1], isoa2 = Part[part, 2]},
 If[Length[Union[Part[part, 1], Part[part, 3]]] == 13, {isob = Part[part, 2],
   isoa1 = Part[part, 1], isoa2 = Part[part, 3]},
  {isob = Part[part, 1], isoa1 = Part[part, 2], isoa2 = Part[part, 3]}
  ]
 ]
If[isob == Part[Part[setsasblocks, 2], 1], {c = Part[Part[sets, 2], 1],
  d1 = Part[Part[sets, 2], 2], d2 = Part[Part[sets, 2], 3]},
 If[isob == Part[Part[setsasblocks, 2], 2], {c = Part[Part[sets, 2], 2],
   d1 = Part[Part[sets, 2], 1], d2 = Part[Part[sets, 2], 3]},
  {c = Part[Part[sets, 2], 3], d1 = Part[Part[sets, 2], 1],
   d2 = Part[Part[sets, 2], 2]}]
 ]
Isotriples = {};
Nonisotriples = {};
If[Length[
   Intersection[Part[Part[perms, 11], c], Part[Part[perms, 1], d1],
    Part[Part[perms, 4], d2]]] >= 1, AppendTo[Isotriples, Part[sets, 2]],
 If[Length[
    Intersection[Part[Part[perms, 11], c], Part[Part[perms, 1], d2],
```

```
    Part[Part[perms, 4], d1]]] >= 1, AppendTo[Isotriples, Part[sets, 2]],
  AppendTo[Nonisotriples, Part[sets, 2]]
  ]
 ]
Isotriples

Out[359]= {{3, 8, 12, 16, 25, 30, 32}, {1, 10, 15, 21, 24, 28, 29}, {1, 11, 14, 22, 23,
  26, 31}}

Out[360]= {15, 1, 4}

Out[363]= {{1, 4, 15}}

Out[364]= {{1, 4, 15}}

In[365]:= Isotriples = {};
Nonisotriples = {};
For[i = 1, i <= Length[sets], i++,
 {part1 = Part[setsasblocks, i],
  part2 = Part[sets, i],
  If[Length[Union[Part[part1, 1], Part[part1, 2]]] ==
    13, {isob = Part[part1, 3], isoa1 = Part[part1, 1],
    isoa2 = Part[part1, 2]},
   If[Length[Union[Part[part1, 1], Part[part1, 3]]] ==
     13, {isob = Part[part1, 2], isoa1 = Part[part1, 1],
     isoa2 = Part[part1, 3]},
    {isob = Part[part1, 1], isoa1 = Part[part1, 2], isoa2 = Part[part1, 3]}
    ]
   ],
  If[isob == Part[part1, 1], {c = Part[part2, 1], d1 = Part[part2, 2],
    d2 = Part[part2, 3]},
   If[isob == Part[part1, 2], {c = Part[part2, 2], d1 = Part[part2, 1],
     d2 = Part[part2, 3]},
    {c = Part[part2, 3], d1 = Part[part2, 1], d2 = Part[part2, 2]}]
   ],
  If[Length[
     Intersection[Part[Part[perms, 11], c], Part[Part[perms, 1], d1],
      Part[Part[perms, 4], d2]]] >= 1, AppendTo[Isotriples, Part[sets, i]],
   If[Length[
      Intersection[Part[Part[perms, 11], c], Part[Part[perms, 1], d2],
```

```
      Part[Part[perms, 4], d1]]] >= 1, AppendTo[Isotriples, Part[sets, i]],
    AppendTo[Nonisotriples, Part[sets, i]]
    ]
   ]
  }
 ]


In[368]:= Isotriples;
Length[Isotriples]

Out[369]= 840

In[370]:= Nonisotriples

Out[370]= {}

In[371]:= k4sets = Subsets[Planes, {4}];
parts = {};
For[i = 1, i <= Length[k4sets], i++,
 {fours = Part[k4sets, i],
  union = Union[Part[fours, 1], Part[fours, 2], Part[fours, 3],
    Part[fours, 4]], If[Length[union] == 26, AppendTo[parts, i]]}
 ]
parts;
sets4 = {};
For[i = 1, i <= Length[parts], i++,
 AppendTo[sets4, Part[k4sets, Part[parts, i]]]]
setsasblocks = sets4;
For[i = 1, i <= 30, i++,
 sets4 = sets4 /. Part[Planes, i] -> i]



In[379]:= Isoquads = {};
NonIsoquads = {};
A1 = 1;
A2 = 4;
B1 = 11;
B2 = 15;
```

```
For[i = 1, i <= Length[sets4], i++,
 {part1 = Part[setsasblocks, i],
  part2 = Part[sets4, i],
  If[Length[Union[Part[part1, 1], Part[part1, 2]]] ==
    13, {a1 = Part[part1, 1], a2 = Part[part1, 2], b1 = Part[part1, 3],
    b2 = Part[part1, 4]},
   If[Length[Union[Part[part1, 1], Part[part1, 3]]] == 13,
    {a1 = Part[part1, 1], a2 = Part[part1, 3], b1 = Part[part1, 2],
     b2 = Part[part1, 4]},
    {a1 = Part[part1, 1], a2 = Part[part1, 4], b1 = Part[part1, 3],
     b2 = Part[part1, 2]}
    ]
   ],
  If[a2 == Part[part1, 2],
   {c1 = Part[part2, 1], c2 = Part[part2, 2], d1 = Part[part2, 3],
    d2 = Part[part2, 4]},
   If[a2 == Part[part1, 3],
    {c1 = Part[part2, 1], c2 = Part[part2, 3], d1 = Part[part2, 2],
     d2 = Part[part2, 4]},
    {c1 = Part[part2, 1], c2 = Part[part2, 4], d1 = Part[part2, 3],
     d2 = Part[part2, 2]}
    ]
   ],
  If[Length[
     Intersection[Part[Part[perms, A1], c1], Part[Part[perms, A2], c2],
      Part[Part[perms, B1], d1], Part[Part[perms, B2], d2]]] >= 1,
   AppendTo[Isoquads, Part[sets, i]],
   If[
    Length[
      Intersection[Part[Part[perms, A1], c2], Part[Part[perms, A2], c1],
       Part[Part[perms, B1], d1], Part[Part[perms, B2], d2]]] >= 1,
    AppendTo[Isoquads, Part[sets, i]],

    ]
   ]
  }
 ]
Length[Isoquads]
Length[NonIsoquads]
```

```
Out[386]= 630


Out[387]= 0


In[388]:= k5sets = Subsets[Planes, {5}];
fivesets = {};
For[i = 1, i <= Length[k5sets], i++,
 {fives = Part[k5sets, i],
  union = Union[Part[fives, 1], Part[fives, 2], Part[fives, 3],
    Part[fives, 4], Part[fives, 5]],
  If[Length[union] == 31, AppendTo[fivesets, i]]}
 ]
fivesets;
sets5 = {};
For[i = 1, i <= Length[fivesets], i++,
 AppendTo[sets5, Part[k5sets, Part[fivesets, i]]]]
setsasblocks5 = sets5;
For[i = 1, i <= 30, i++,
 sets5 = sets5 /. Part[Planes, i] -> i]


In[396]:= Isoquints = {};
NonIsoquints = {};
A1 = 1;
A2 = 4;
A3 = 8;
B1 = 15;
B2 = 26;
For[i = 1, i <= Length[sets5], i++,
 {part1 = Part[setsasblocks5, i],
  part2 = Part[sets5, i],
  possibleA = Subsets[part1, {3}],
  For[j = 1, j <= Length[possibleA], j++, {
    check = Part[possibleA, j],
    If[Length[Intersection[Part[check, 1], Part[check, 2]]] == 1 &&
      Length[Intersection[Part[check, 1], Part[check, 3]]] == 1 &&
      Length[Intersection[Part[check, 3], Part[check, 2]]] == 1,
     {a1 = Part[check, 1],
      a2 = Part[check, 2],
```

```
      a3 = Part[check, 3],
      possibleB = Complement[part1, check],
      b1 = Part[possibleB, 1],
      b2 = Part[possibleB, 2],
      AB = {a1, a2, a3, b1, b2},
      For[k = 1, k <= 30, k++,
       AB = AB /. Part[Planes, k] -> k
       ],
      c1 = Part[AB, 1],
      c2 = Part[AB, 2],
      c3 = Part[AB, 3],
      d1 = Part[AB, 4],
      d2 = Part[AB, 5],

      j = 11}
     ]
    }
   ],
  If[Length[
     Intersection[Part[Part[perms, A1], c1], Part[Part[perms, A2], c2],
      Part[Part[perms, A3], c3], Part[Part[perms, B1], d1],
      Part[Part[perms, B2], d2]]] >= 1, AppendTo[Isoquints, Part[sets5, i]],
   If[Length[
      Intersection[Part[Part[perms, A1], c1], Part[Part[perms, A2], c2],
       Part[Part[perms, A3], c3], Part[Part[perms, B1], d2],
       Part[Part[perms, B2], d1]]] >= 1, AppendTo[Isoquints, Part[sets5, i]]

    ]
   ]




  }

 ]
Length[Isoquints]
```

```
Out[404]= 840

In[405]:= k6sets = Subsets[Planes, {6}];
sixsets = {};
For[i = 1, i <= Length[k6sets], i++,
 {six = Part[k6sets, i],
  union = Union[Part[six, 1], Part[six, 2], Part[six, 3], Part[six, 4],
    Part[six, 5], Part[six, 6]],
  If[Length[union] == 35, AppendTo[sixsets, i]]}
 ]

sets6 = {};
For[i = 1, i <= Length[sixsets], i++,
 AppendTo[sets6, Part[k6sets, Part[sixsets, i]]]]
setsasblocks6 = sets6;
For[i = 1, i <= 30, i++,
 sets6 = sets6 /. Part[Planes, i] -> i]

In[412]:= Part[sets6, 1]

Out[412]= {1, 4, 8, 15, 24, 26}

In[413]:= fourA = {};
For[i = 1, i <= Length[setsasblocks6], i++,
 {part1 = Part[setsasblocks6, i],
  part2 = Part[sets6, i],
  possibleA = Subsets[part1, {4}],
  For[j = 1, j <= Length[possibleA], j++,
   {check = Part[possibleA, j],
    If[Length[Intersection[Part[check, 1], Part[check, 2]]] == 1 &&
      Length[Intersection[Part[check, 1], Part[check, 3]]] == 1 &&
      Length[Intersection[Part[check, 1], Part[check, 4]]] == 1,
     {AppendTo[fourA, part2], j = Length[possibleA] + 1}
     ]
    }
   ]
  }
 ]
```

```
In[415]:= Length[setsasblocks6]
Length[fourA]

Out[415]= 210

Out[416]= 210

In[417]:= Part[sets6, 1]
Part[setsasblocks6, 1]

Out[417]= {1, 4, 8, 15, 24, 26}

Out[418]= {{1, 10, 15, 21, 24, 28, 29}, {1, 11, 14, 22, 23, 26, 31}, {2, 7, 15, 19, 23,
  27, 33}, {3, 8, 12, 16, 25, 30, 32}, {4, 9, 10, 17, 22, 27, 35}, {5, 6, 13,
  18, 20, 30, 34}}

In[419]:= Isosix = {};
NonIsosix = {};
A1 = 1;
A2 = 4;
A3 = 8;
A4 = 24;
B1 = 15;
B2 = 26;
For[i = 1, i <= Length[sets6], i++,
 {part1 = Part[setsasblocks6, i],
  part2 = Part[sets6, i],
  possibleA = Subsets[part1, {4}],
  For[j = 1, j <= Length[possibleA], j++,
   {check = Part[possibleA, j],
    If[Length[Intersection[Part[check, 1], Part[check, 2]]] == 1 &&
      Length[Intersection[Part[check, 1], Part[check, 3]]] == 1 &&
      Length[Intersection[Part[check, 1], Part[check, 4]]] == 1,
     {a1 = Part[check, 1],
      a2 = Part[check, 2],
      a3 = Part[check, 3],
      a4 = Part[check, 4],
      possibleB = Complement[part1, check],
      b1 = Part[possibleB, 1],
      b2 = Part[possibleB, 2],
```

```
      AB = {a1, a2, a3, a4, b1, b2},
      For[k = 1, k <= 30, k++,
       AB = AB /. Part[Planes, k] -> k
       ],
      c1 = Part[AB, 1],
      c2 = Part[AB, 2],
      c3 = Part[AB, 3],
      c4 = Part[AB, 4],
      d1 = Part[AB, 5],
      d2 = Part[AB, 6],
      j = Length[possibleA] + 1}
     ]
    }
   ], If[Length[
     Intersection[Part[Part[perms, A1], c1], Part[Part[perms, A2], c2],
      Part[Part[perms, A3], c3], Part[Part[perms, A4, c4]],
      Part[Part[perms, B1], d1], Part[Part[perms, B2], d2]]] >= 1,
   AppendTo[Isosix, Part[sets6, i]],
   If[Length[
      Intersection[Part[Part[perms, A1], c1], Part[Part[perms, A2], c2],
       Part[Part[perms, A3], c3], Part[Part[perms, A4, c4]],
       Part[Part[perms, B1], d2], Part[Part[perms, B2], d1]]] >= 1,
    AppendTo[Isosix, Part[sets6, i]]]
   ]
  }
 ]
Length[Isosix]

Out[428]= 210
```

# D *Mathematica* Code: Large Set of (11,5,2) Designs

```
First initialize the Combinatorica package.

<< Combinatorica`

Define the base design and the permutations used.  The permutations labeled "perm1-perm6"
 correspond to the six permutations that act on the base design to create the six starter
  designs.  The permutation labeled "perm" is the permutation used to create the group
  that acts on each starter design to create the large set.

Design = {{1, 2, 3, 7, 10}, {1, 2, 6, 9, 11}, {1, 3, 4, 5, 9}, {1, 4, 6, 7,
    8}, {1, 5, 8, 10, 11}, {2, 3, 4, 8, 11}, {2, 4, 5, 6, 10}, {2, 5, 7, 8,
    9}, {3, 5, 6, 7, 11}, {3, 6, 8, 9, 10}, {4, 7, 9, 10, 11}};
perm1 = {1 -> 2, 2 -> 5, 5 -> 4, 4 -> 6, 6 -> 10, 10 -> 8, 8 -> 3, 3 -> 7,
   7 -> 9, 9 -> 11, 11 -> 1};
perm2 = {1 -> 4, 4 -> 8, 8 -> 11, 11 -> 10, 10 -> 5, 5 -> 7, 7 -> 1, 2 -> 6,
   6 -> 2, 3 -> 9, 9 -> 3};
perm3 = {6 -> 11, 11 -> 7, 7 -> 10, 10 -> 8, 8 -> 6};
perm4 = {1 -> 3, 3 -> 2, 2 -> 8, 8 -> 4, 4 -> 6, 6 -> 1, 5 -> 7, 7 -> 5};
perm5 = {6 -> 8, 8 -> 9, 9 -> 7, 7 -> 10, 10 -> 6};
perm6 = {1 -> 3, 3 -> 11, 11 -> 1, 2 -> 9, 9 -> 5, 5 -> 7, 7 -> 10, 10 -> 2,
   6 -> 8, 8 -> 6};
perm = {5 -> 6, 6 -> 7, 7 -> 8, 8 -> 9, 9 -> 10, 10 -> 11, 11 -> 5};

Create the six starter designs.

A0 = Design /. perm1
B0 = Design /. perm2
C0 = Design /. perm3
D0 = Design /. perm4
E0 = Design /. perm5
F0 = Design /. perm6

{{2, 5, 7, 9, 8}, {2, 5, 10, 11, 1}, {2, 7, 6, 4, 11}, {2, 6, 10, 9, 3}, {2,
  4, 3, 8, 1}, {5, 7, 6, 3, 1}, {5, 6, 4, 10, 8}, {5, 4, 9, 3, 11}, {7, 4, 10,
   9, 1}, {7, 10, 3, 11, 8}, {6, 9, 11, 8, 1}}
```

```
{{4, 6, 9, 1, 5}, {4, 6, 2, 3, 10}, {4, 9, 8, 7, 3}, {4, 8, 2, 1, 11}, {4, 7,
  11, 5, 10}, {6, 9, 8, 11, 10}, {6, 8, 7, 2, 5}, {6, 7, 1, 11, 3}, {9, 7, 2,
  1, 10}, {9, 2, 11, 3, 5}, {8, 1, 3, 5, 10}}

{{1, 2, 3, 10, 8}, {1, 2, 11, 9, 7}, {1, 3, 4, 5, 9}, {1, 4, 11, 10, 6}, {1,
  5, 6, 8, 7}, {2, 3, 4, 6, 7}, {2, 4, 5, 11, 8}, {2, 5, 10, 6, 9}, {3, 5, 11,
   10, 7}, {3, 11, 6, 9, 8}, {4, 10, 9, 8, 7}}

{{3, 8, 2, 5, 10}, {3, 8, 1, 9, 11}, {3, 2, 6, 7, 9}, {3, 6, 1, 5, 4}, {3, 7,
  4, 10, 11}, {8, 2, 6, 4, 11}, {8, 6, 7, 1, 10}, {8, 7, 5, 4, 9}, {2, 7, 1,
  5, 11}, {2, 1, 4, 9, 10}, {6, 5, 9, 10, 11}}

{{1, 2, 3, 10, 6}, {1, 2, 8, 7, 11}, {1, 3, 4, 5, 7}, {1, 4, 8, 10, 9}, {1, 5,
   9, 6, 11}, {2, 3, 4, 9, 11}, {2, 4, 5, 8, 6}, {2, 5, 10, 9, 7}, {3, 5, 8,
  10, 11}, {3, 8, 9, 7, 6}, {4, 10, 7, 6, 11}}

{{3, 9, 11, 10, 2}, {3, 9, 8, 5, 1}, {3, 11, 4, 7, 5}, {3, 4, 8, 10, 6}, {3,
  7, 6, 2, 1}, {9, 11, 4, 6, 1}, {9, 4, 7, 8, 2}, {9, 7, 10, 6, 5}, {11, 7, 8,
   10, 1}, {11, 8, 6, 5, 2}, {4, 10, 5, 2, 1}}

Let the group generated by "perm" act on each base design.  The lists perm1list,
 perm2list, perm3list, perm4list, perm5list, and perm6list store the seven designs
  created by having the group generated by "perm" act on the corresponding starter
   design.

perm1list = {};
For[i = 1, i <= 7, i++,
 {A0 = A0 /. perm,
  AppendTo[perm1list, A0]}]

perm2list = {};
For[i = 1, i <= 7, i++,
 {B0 = B0 /. perm,
  AppendTo[perm2list, B0]}]

perm3list = {};
For[i = 1, i <= 7, i++,
 {C0 = C0 /. perm,
  AppendTo[perm3list, C0]}]
```

```
perm4list = {};
For[i = 1, i <= 7, i++,
 {D0 = D0 /. perm,
  AppendTo[perm4list, D0]}]

perm5list = {};
For[i = 1, i <= 7, i++,
 {E0 = E0 /. perm,
  AppendTo[perm5list, E0]}]

perm6list = {};
For[i = 1, i <= 7, i++,
 {F0 = F0 /. perm,
  AppendTo[perm6list, F0]}]



perm1list

{{{2, 6, 8, 10, 9}, {2, 6, 11, 5, 1}, {2, 8, 7, 4, 5}, {2, 7, 11, 10, 3},
{2, 4, 3, 9, 1}, {6, 8, 7, 3, 1}, {6, 7, 4, 11, 9}, {6, 4, 10, 3, 5},
{8, 4,11, 10, 1}, {8, 11, 3, 5, 9}, {7, 10, 5, 9, 1}}, {{2, 7, 9, 11, 10},
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perm2list

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perm4list

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perm5list

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```

perm6list

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{4, 10, 5, 2, 1}}}
```

Define largeset to be the union of the above 6 lists.  The length of this list is 42.

```
largeset =
  Union[perm1list, perm2list, perm3list, perm4list, perm5list, perm6list];
Length[largeset]
```

```
42
```

```
The largeset is "flattened" into a list of just the blocks and each block is sorted.
The union command eliminates any repeated blocks.  The length of this list is 42*11,
which confirms that every block is unique.
```

```
largeset;
blocks = Flatten[largeset, 1];
sortedblocks = {};
For[i = 1, i <= Length[blocks], i++,
 {part = Part[blocks, i],
  part = Sort[part],
  AppendTo[sortedblocks, part]
  }
 ]
```

```
sortedblocks;
list = Union[sortedblocks];
Length[list]
42*11
list
```

```
462
```

```
462
```

```
{{1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 4, 7}, {1, 2, 3, 4, 8},
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{2, 3, 5, 7, 8}, {2, 3, 5, 7, 9}, {2, 3, 5, 7, 10}, {2, 3, 5, 7, 11},
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{2, 4, 5, 10, 11}, {2, 4, 6, 7, 8}, {2, 4, 6, 7, 9}, {2, 4, 6, 7, 10},
{2, 4, 6, 7, 11}, {2, 4, 6, 8, 9}, {2, 4, 6, 8, 10}, {2, 4, 6, 8, 11},
{2, 4, 6, 9, 10}, {2, 4, 6, 9, 11}, {2, 4, 6, 10, 11}, {2, 4, 7, 8, 9},
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{2, 5, 6, 7, 11}, {2, 5, 6, 8, 9}, {2, 5, 6, 8, 10}, {2, 5, 6, 8, 11},
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{3, 4, 5, 6, 11}, {3, 4, 5, 7, 8}, {3, 4, 5, 7, 9}, {3, 4, 5, 7, 10},
{3, 4, 5, 7, 11}, {3, 4, 5, 8, 9}, {3, 4, 5, 8, 10}, {3, 4, 5, 8, 11},
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{3, 4, 8, 9, 11}, {3, 4, 8, 10, 11}, {3, 4, 9, 10, 11}, {3, 5, 6, 7, 8},
{3, 5, 6, 7, 9}, {3, 5, 6, 7, 10}, {3, 5, 6, 7, 11}, {3, 5, 6, 8, 9},
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{3, 5, 7, 9, 10}, {3, 5, 7, 9, 11}, {3, 5, 7, 10, 11}, {3, 5, 8, 9, 10},
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{4, 5, 7, 8, 9}, {4, 5, 7, 8, 10}, {4, 5, 7, 8, 11}, {4, 5, 7, 9, 10},
{4, 5, 7, 9, 11}, {4, 5, 7, 10, 11}, {4, 5, 8, 9, 10}, {4, 5, 8, 9, 11},
{4, 5, 8, 10, 11}, {4, 5, 9, 10, 11}, {4, 6, 7, 8, 9}, {4, 6, 7, 8, 10},
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{5, 6, 7, 9, 10}, {5, 6, 7, 9, 11}, {5, 6, 7, 10, 11}, {5, 6, 8, 9, 10},
{5, 6, 8, 9, 11}, {5, 6, 8, 10, 11}, {5, 6, 9, 10, 11}, {5, 7, 8, 9, 10},
{5, 7, 8, 9, 11}, {5, 7, 8, 10, 11}, {5, 7, 9, 10, 11}, {5, 8, 9, 10, 11},
{6, 7, 8, 9, 10}, {6, 7, 8, 9, 11}, {6, 7, 8, 10, 11}, {6, 7, 9, 10, 11},
{6, 8, 9, 10, 11}, {7, 8, 9, 10, 11}}

# E Complete Solution of Marco Buratti's Problem for $p = 7$

We first include a table that includes every possible set $S$ of lengths in a Hamiltonian path along with a Hamiltonian path in $K_7$ whose edge lengths correspond to the values in the multi-set $S$.

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{1,1,1,1,1,1\}$ | $\{0,1,2,3,4,5,6\}$ | $\{1,1,3,3,3,3\}$ | $\{0,3,2,6,5,1,4\}$ |
| $\{1,1,1,1,1,2\}$ | $\{0,1,2,3,4,6,5\}$ | $\{1,2,2,2,2,2\}$ | $\{0,2,4,5,3,1,6\}$ |
| $\{1,1,1,1,1,3\}$ | $\{0,1,2,3,6,5,4\}$ | $\{1,2,2,2,2,3\}$ | $\{0,2,4,1,6,5,3\}$ |
| $\{1,1,1,1,2,2\}$ | $\{0,1,2,3,5,4,6\}$ | $\{1,2,2,2,3,3\}$ | $\{0,2,5,6,4,1,3\}$ |
| $\{1,1,1,1,2,3\}$ | $\{0,1,2,3,6,4,5\}$ | $\{1,2,2,3,3,3\}$ | $\{0,1,4,2,5,3,6\}$ |
| $\{1,1,1,1,3,3\}$ | $\{0,1,2,5,6,3,4\}$ | $\{1,2,3,3,3,3\}$ | $\{0,3,6,4,1,2,5\}$ |
| $\{1,1,1,2,2,2\}$ | $\{0,1,2,4,6,5,3\}$ | $\{1,3,3,3,3,3\}$ | $\{0,3,4,1,5,2,6\}$ |
| $\{1,1,1,2,2,3\}$ | $\{0,1,2,6,4,5,3\}$ | $\{2,2,2,2,2,2\}$ | $\{0,2,4,6,1,3,5\}$ |
| $\{1,1,1,2,3,3\}$ | $\{0,1,2,5,4,6,3\}$ | $\{2,2,2,2,2,3\}$ | $\{0,2,4,6,1,5,3\}$ |
| $\{1,1,1,3,3,3\}$ | $\{0,1,4,5,2,3,6\}$ | $\{2,2,2,2,3,3\}$ | $\{0,2,4,6,3,1,5\}$ |
| $\{1,1,2,2,2,2\}$ | $\{0,1,6,4,2,3,5\}$ | $\{2,2,2,3,3,3\}$ | $\{0,2,5,3,6,4,1\}$ |
| $\{1,1,2,2,2,3\}$ | $\{0,2,3,5,6,1,4\}$ | $\{2,2,3,3,3,3\}$ | $\{0,3,5,2,6,1,4\}$ |
| $\{1,1,2,2,3,3\}$ | $\{0,1,3,6,4,5,2\}$ | $\{2,3,3,3,3,3\}$ | $\{0,3,6,4,1,5,2\}$ |
| $\{1,1,2,3,3,3\}$ | $\{0,3,4,1,6,5,2\}$ | $\{3,3,3,3,3,3\}$ | $\{0,3,6,2,5,1,4\}$ |

Included next is the *Mathematica* code used to determine that all sets $S$ for $p = 7$ are realizable.

```
Find all permutations on 7 elements.

a = {0, 1, 2, 3, 4, 5, 6};
perms = Permutations[a];

For each permutation of 7 elements, calculate the set of differences between consecutive
 elements.

diff = {};
For[j = 1, j <= Length[perms], j++,
 {diffs = {0, 0, 0, 0, 0, 0},
  perm = Part[perms, j],
  For[i = 1, i <= 6, i++,
   {value = Part[perm, i + 1] - Part[perm, i],
    Part[diffs, i] = value
    }
   ],
  AppendTo[diff, diffs]
  }
 ]

Replace the elements in each set with the corresponding element between 1 and (p-1)/2.

diff = diff /. -1 -> 1;
diff = diff /. -2 -> 2;
diff = diff /. -3 -> 3;
diff = diff /. -4 -> 3;
diff = diff /. -5 -> 2;
diff = diff /. -6 -> 1;
diff = diff /. 4 -> 3;
diff = diff /. 5 -> 2;
diff = diff /. 6 -> 1;

Sort each element.  The union command removes any repeated elements.

diff = Union[diff];
newlist = {};
For[i = 1, i <= Length[diff], i++,
 {part = Sort[Part[diff, i]],
```

```
  AppendTo[newlist, part]
  }
 ]
Union[newlist]
```

```
{{1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 2}, {1, 1, 1, 1, 1, 3},
{1, 1, 1, 1, 2, 2}, {1, 1, 1, 1, 2, 3}, {1, 1, 1, 1, 3, 3},
{1, 1, 1, 2, 2, 2}, {1, 1, 1, 2, 2, 3}, {1, 1, 1, 2, 3, 3},
{1, 1, 1, 3, 3, 3}, {1, 1, 2, 2, 2, 2}, {1, 1, 2, 2, 2, 3},
{1, 1, 2, 2, 3, 3}, {1, 1, 2, 3, 3, 3}, {1, 1, 3, 3, 3, 3},
{1, 2, 2, 2, 2, 2}, {1, 2, 2, 2, 2, 3}, {1, 2, 2, 2, 3, 3},
{1, 2, 2, 3, 3, 3}, {1, 2, 3, 3, 3, 3}, {1, 3, 3, 3, 3, 3},
{2, 2, 2, 2, 2, 2}, {2, 2, 2, 2, 2, 3}, {2, 2, 2, 2, 3, 3},
{2, 2, 2, 3, 3, 3}, {2, 2, 3, 3, 3, 3}, {2, 3, 3, 3, 3,3},
{3, 3, 3, 3, 3, 3}}
```

# F Complete Solution of Marco Buratti's Problem for $s = 8$

We first include a table that includes every possible set $S$ of lengths in a Hamiltonian path along with a Hamiltonian path in $K_8$ whose edge lengths correspond to the values in the multi-set $S$ when a Hamiltonian path exists.

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{1,1,1,1,1,1,1\}$ | $\{0,1,2,3,4,5,6,7\}$ | $\{1,1,1,1,1,1,2\}$ | $\{0,1,2,3,4,5,7,6\}$ |
| $\{1,1,1,1,1,1,3\}$ | $\{0,1,2,3,4,7,6,5\}$ | $\{1,1,1,1,1,1,4\}$ | $\{0,1,2,3,7,6,5,4\}$ |
| $\{1,1,1,1,1,2,2\}$ | $\{0,1,2,3,4,6,5,7\}$ | $\{1,1,1,1,1,2,3\}$ | $\{0,1,2,3,4,7,5,6\}$ |
| $\{1,1,1,1,1,2,4\}$ | $\{0,1,2,3,7,6,4,5\}$ | $\{1,1,1,1,1,3,3\}$ | $\{0,1,2,3,6,5,4,7\}$ |
| $\{1,1,1,1,1,3,4\}$ | $\{0,1,2,3,7,4,5,6\}$ | $\{1,1,1,1,1,4,4\}$ | $\{0,1,2,6,5,4,3,7\}$ |
| $\{1,1,1,1,2,2,2\}$ | $\{0,1,2,3,5,7,6,4\}$ | $\{1,1,1,1,2,2,3\}$ | $\{0,1,2,3,5,6,4,7\}$ |
| $\{1,1,1,1,2,2,4\}$ | $\{0,1,2,3,7,5,4,6\}$ | $\{1,1,1,1,2,3,3\}$ | $\{0,1,2,3,6,5,7,4\}$ |
| $\{1,1,1,1,2,3,4\}$ | $\{0,1,2,3,7,4,6,5\}$ | $\{1,1,1,1,2,4,4\}$ | $\{0,1,2,6,5,7,3,4\}$ |
| $\{1,1,1,1,3,3,3\}$ | $\{0,1,2,5,4,7,6,3\}$ | $\{1,1,1,1,3,3,4\}$ | $\{0,1,2,7,3,6,5,4\}$ |
| $\{1,1,1,1,3,4,4\}$ | $\{0,1,2,6,5,4,7,3\}$ | $\{1,1,1,1,4,4,4\}$ | $\{0,1,5,4,3,7,6,2\}$ |
| $\{1,1,1,2,2,2,2\}$ | $\{0,1,2,4,6,7,5,3\}$ | $\{1,1,1,2,2,2,3\}$ | $\{0,1,2,3,5,7,4,6\}$ |
| $\{1,1,1,2,2,2,4\}$ | $\{0,1,2,4,6,5,3,7\}$ | $\{1,1,1,2,2,3,3\}$ | $\{0,1,2,3,6,4,7,5\}$ |
| $\{1,1,1,2,2,3,4\}$ | $\{0,1,2,4,5,7,3,6\}$ | $\{1,1,1,2,2,4,4\}$ | $\{0,1,2,6,4,3,7,5\}$ |
| $\{1,1,1,2,3,3,3\}$ | $\{0,1,2,5,3,6,7,4\}$ | $\{1,1,1,2,3,3,4\}$ | $\{0,1,2,4,7,3,6,5\}$ |
| $\{1,1,1,2,3,4,4\}$ | $\{0,1,2,6,3,7,5,4\}$ | $\{1,1,1,2,4,4,4\}$ | $\{0,1,5,4,2,6,7,3\}$ |
| $\{1,1,1,3,3,3,3\}$ | $\{0,1,4,3,6,5,2,7\}$ | $\{1,1,1,3,3,3,4\}$ | $\{0,1,2,5,4,7,3,6\}$ |
| $\{1,1,1,3,3,4,4\}$ | $\{0,1,2,6,3,7,4,5\}$ | $\{1,1,1,3,4,4,4\}$ | $\{0,1,5,2,6,7,3,4\}$ |
| $\{1,1,1,4,4,4,4\}$ | $\{0,4,3,7,6,2,1,5\}$ | $\{1,1,2,2,2,2,2\}$ | $\{0,6,4,3,5,7,1,2\}$ |
| $\{1,1,2,2,2,2,3\}$ | $\{0,1,2,4,6,3,5,7\}$ | $\{1,1,2,2,2,2,4\}$ | $\{0,6,4,5,3,7,1,2\}$ |
| $\{1,1,2,2,2,3,3\}$ | $\{0,1,2,4,7,5,3,6\}$ | $\{1,1,2,2,2,3,4\}$ | $\{0,1,2,4,6,3,7,5\}$ |
| $\{1,1,2,2,2,4,4\}$ | $\{0,6,5,1,7,3,2,4\}$ | $\{1,1,2,2,3,3,3\}$ | $\{0,1,2,5,3,6,4,7\}$ |
| $\{1,1,2,2,3,3,4\}$ | $\{0,1,2,5,3,7,4,6\}$ | $\{1,1,2,2,3,4,4\}$ | $\{0,1,2,6,4,7,3,5\}$ |
| $\{1,1,2,2,4,4,4\}$ | $\{0,7,3,1,5,4,2,6\}$ | $\{1,1,2,3,3,3,3\}$ | $\{0,7,4,1,2,5,3,6\}$ |
| $\{1,1,2,3,3,3,4\}$ | $\{0,7,4,1,2,6,3,5\}$ | $\{1,1,2,3,3,4,4\}$ | $\{0,7,4,1,5,3,2,6\}$ |
| $\{1,1,2,3,4,4,4\}$ | $\{0,1,5,2,6,4,3,7\}$ | $\{1,1,2,4,4,4,4\}$ | $\{0,4,3,7,1,5,6,2\}$ |
| $\{1,1,3,3,3,3,3\}$ | $\{0,5,2,1,4,7,6,3\}$ | $\{1,1,3,3,3,3,4\}$ | $\{0,5,2,3,7,6,1,4\}$ |

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{1,1,3,3,3,4,4\}$ | $\{0,5,2,6,7,3,4,1\}$ | $\{1,1,3,3,4,4,4\}$ | $\{0,7,3,4,1,5,2,6\}$ |
| $\{1,1,3,4,4,4,4\}$ | $\{0,4,1,5,6,2,3,7\}$ | $\{1,1,4,4,4,4,4\}$ | $DNE$ |
| $\{1,2,2,2,2,2,2\}$ | $\{0,6,4,2,1,3,5,7\}$ | $\{1,2,2,2,2,2,3\}$ | $\{0,6,4,3,1,7,5,2\}$ |
| $\{1,2,2,2,2,2,4\}$ | $\{0,2,4,6,5,3,7,1\}$ | $\{1,2,2,2,2,3,3\}$ | $\{0,2,4,7,1,3,6,5\}$ |
| $\{1,2,2,2,2,3,4\}$ | $\{0,2,6,1,7,5,3,4\}$ | $\{1,2,2,2,2,4,4\}$ | $\{0,2,6,4,3,1,5,7\}$ |
| $\{1,2,2,2,3,3,3\}$ | $\{0,2,7,1,4,5,3,6\}$ | $\{1,2,2,2,3,3,4\}$ | $\{0,2,7,1,5,4,6,3\}$ |
| $\{1,2,2,2,3,4,4\}$ | $\{0,2,7,3,1,5,4,6\}$ | $\{1,2,2,2,4,4,4\}$ | $\{0,4,2,6,5,3,7,1\}$ |
| $\{1,2,2,3,3,3,3\}$ | $\{0,5,2,1,3,6,4,7\}$ | $\{1,2,2,3,3,3,4\}$ | $\{0,5,2,1,3,7,4,6\}$ |
| $\{1,2,2,3,3,4,4\}$ | $\{0,5,2,6,4,3,7,1\}$ | $\{1,2,2,3,4,4,4\}$ | $\{0,6,2,1,5,3,7,4\}$ |
| $\{1,2,2,4,4,4,4\}$ | $\{0,4,2,6,5,1,3,7\}$ | $\{1,2,3,3,3,3,3\}$ | $\{0,5,2,3,6,1,7,4\}$ |
| $\{1,2,3,3,3,3,4\}$ | $\{0,5,2,3,7,4,6,1\}$ | $\{1,2,3,3,3,4,4\}$ | $\{0,5,2,6,1,7,3,4\}$ |
| $\{1,2,3,3,4,4,4\}$ | $\{0,6,2,3,7,4,1,5\}$ | $\{1,2,3,4,4,4,4\}$ | $\{0,4,1,5,3,7,6,2\}$ |
| $\{1,2,4,4,4,4,4\}$ | $DNE$ | $\{1,3,3,3,3,3,3\}$ | $\{0,5,2,3,6,1,4,7\}$ |
| $\{1,3,3,3,3,3,4\}$ | $\{0,5,2,3,7,4,1,6\}$ | $\{1,3,3,3,3,4,4\}$ | $\{0,5,2,6,1,4,3,7\}$ |
| $\{1,3,3,3,4,4,4\}$ | $\{0,7,3,6,2,5,1,4\}$ | $\{1,3,3,4,4,4,4\}$ | $\{0,4,1,5,2,6,7,3\}$ |
| $\{1,3,4,4,4,4,4\}$ | $DNE$ | $\{1,4,4,4,4,4,4\}$ | $DNE$ |
| $\{2,2,2,2,2,2,2\}$ | $DNE$ | $\{2,2,2,2,2,2,3\}$ | $\{0,6,4,2,5,3,1,7\}$ |
| $\{2,2,2,2,2,2,4\}$ | $DNE$ | $\{2,2,2,2,2,3,3\}$ | $\{0,6,4,7,1,3,5,2\}$ |
| $\{2,2,2,2,2,3,4\}$ | $\{0,2,4,6,1,3,7,5\}$ | $\{2,2,2,2,2,4,4\}$ | $DNE$ |
| $\{2,2,2,2,3,3,3\}$ | $\{0,2,4,7,1,6,3,5\}$ | $\{2,2,2,2,3,3,4\}$ | $\{0,2,4,7,5,1,3,6\}$ |
| $\{2,2,2,2,3,4,4\}$ | $\{0,2,6,4,1,3,7,5\}$ | $\{2,2,2,2,4,4,4\}$ | $DNE$ |
| $\{2,2,2,3,3,3,3\}$ | $\{0,2,7,5,3,6,1,4\}$ | $\{2,2,2,3,3,3,4\}$ | $\{0,3,1,4,6,2,5,7\}$ |
| $\{2,2,2,3,3,4,4\}$ | $\{0,3,1,5,7,2,6,4\}$ | $\{2,2,2,3,4,4,4\}$ | $\{0,4,2,6,1,3,7,5\}$ |
| $\{2,2,2,4,4,4,4\}$ | $DNE$ | $\{2,2,3,3,3,3,3\}$ | $\{0,5,2,4,7,1,6,3\}$ |
| $\{2,2,3,3,3,3,4\}$ | $\{0,5,2,4,7,3,1,6\}$ | $\{2,2,3,3,3,4,4\}$ | $\{0,5,2,6,4,1,3,7\}$ |
| $\{2,2,3,3,4,4,4\}$ | $\{0,6,2,5,1,3,7,4\}$ | $\{2,2,3,4,4,4,4\}$ | $\{0,4,2,6,1,5,3,7\}$ |
| $\{2,2,4,4,4,4,4\}$ | $DNE$ | $\{2,3,3,3,3,3,3\}$ | $\{0,5,2,7,4,1,3,6\}$ |
| $\{2,3,3,3,3,3,4\}$ | $\{0,5,3,6,2,7,4,1\}$ | $\{2,3,3,3,3,4,4\}$ | $\{0,5,3,7,2,6,1,4\}$ |
| $\{2,3,3,3,4,4,4\}$ | $\{0,6,2,5,1,4,7,3\}$ | $\{2,3,3,4,4,4,4\}$ | $\{0,4,1,5,3,7,2,6\}$ |
| $\{2,3,4,4,4,4,4\}$ | $DNE$ | $\{2,4,4,4,4,4,4\}$ | $DNE$ |
| $\{3,3,3,3,3,3,3\}$ | $\{0,5,2,7,4,1,6,3\}$ | $\{3,3,3,3,3,3,4\}$ | $\{0,3,6,1,5,2,7,4\}$ |
| $\{3,3,3,3,3,4,4\}$ | $\{0,3,6,2,5,1,4,7\}$ | $\{3,3,3,3,4,4,4\}$ | $\{0,3,7,4,1,5,2,6\}$ |
| $\{3,3,3,4,4,4,4\}$ | $\{0,4,1,5,2,6,3,7\}$ | $\{3,3,4,4,4,4,4\}$ | $DNE$ |
| $\{3,4,4,4,4,4,4\}$ | $DNE$ | $\{4,4,4,4,4,4,4\}$ | $DNE$ |

Included next is the *Mathematica* code used to determine which sets $S$ for $s = 8$ are realizable.

```
Find all permutations on 8 elements.

a = {1, 2, 3, 4, 5, 6, 7, 8};
perms = Permutations[a];
Length[perms]

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Determine the possible sets S by calculating the difference between consecutive elements in each permutation.


diff = {};
For[j = 1, j <= Length[perms], j++,
 {diffs = {0, 0, 0, 0, 0, 0, 0},
  perm = Part[perms, j],
  For[i = 1, i <= 7, i++,
   {value = Part[perm, i + 1] - Part[perm, i],
    Part[diffs, i] = value
    }
   ],
  AppendTo[diff, diffs]
  }
 ]


Reduce the elements in the path to elements between 0 and 4.

diff = diff /. -1 -> 1;
diff = diff /. -2 -> 2;
diff = diff /. -3 -> 3;
diff = diff /. -4 -> 4;
diff = diff /. -5 -> 5;
diff = diff /. -6 -> 6;
diff = diff /. -7 -> 7;
diff = diff /. 5 -> 3;
diff = diff /. 6 -> 2;
```

```
diff = diff /. 7 -> 1;
```

Sort each element and remove repeat elements

```
diff = Union[diff];
newlist = {};
For[k = 1, k <= Length[diff], k++,
 {part = Sort[Part[diff, k]],
  AppendTo[newlist, part]
  }
 ]
newlist = Union[newlist]
Length[newlist]
```

```
{{1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 2}, {1, 1, 1, 1, 1, 1, 3},
{1, 1, 1, 1, 1, 1, 4}, {1, 1, 1, 1, 1, 2, 2}, {1, 1, 1, 1, 1, 2, 3},
{1, 1, 1, 1, 1, 2, 4}, {1, 1, 1, 1, 1, 3, 3}, {1, 1, 1, 1, 1, 3, 4},
{1, 1, 1, 1, 1, 4, 4}, {1, 1, 1, 1, 2, 2, 2}, {1, 1, 1, 1, 2, 2, 3},
{1, 1, 1, 1, 2, 2, 4}, {1, 1, 1, 1, 2, 3, 3}, {1, 1, 1, 1, 2, 3, 4},
{1, 1, 1, 1, 2, 4, 4}, {1, 1, 1, 1, 3, 3, 3}, {1, 1, 1, 1, 3, 3, 4},
{1, 1, 1, 1, 3, 4, 4}, {1, 1, 1, 1, 4, 4, 4}, {1, 1, 1, 2, 2, 2, 2},
{1, 1, 1, 2, 2, 2, 3}, {1, 1, 1, 2, 2, 2, 4}, {1, 1, 1, 2, 2, 3, 3},
{1, 1, 1, 2, 2, 3, 4}, {1, 1, 1, 2, 2, 4, 4}, {1, 1, 1, 2, 3, 3, 3},
{1, 1, 1, 2, 3, 3, 4}, {1, 1, 1, 2, 3, 4, 4}, {1, 1, 1, 2, 4, 4, 4},
{1, 1, 1, 3, 3, 3, 3}, {1, 1, 1, 3, 3, 3, 4}, {1, 1, 1, 3, 3, 4, 4},
{1, 1, 1, 3, 4, 4, 4}, {1, 1, 1, 4, 4, 4, 4}, {1, 1, 2, 2, 2, 2, 2},
{1, 1, 2, 2, 2, 2, 3}, {1, 1, 2, 2, 2, 2, 4}, {1, 1, 2, 2, 2, 3, 3},
{1, 1, 2, 2, 2, 3, 4}, {1, 1, 2, 2, 2, 4, 4}, {1, 1, 2, 2, 3, 3, 3},
{1, 1, 2, 2, 3, 3, 4}, {1, 1, 2, 2, 3, 4, 4}, {1, 1, 2, 2, 4, 4, 4},
{1, 1, 2, 3, 3, 3, 3}, {1, 1, 2, 3, 3, 3, 4}, {1, 1, 2, 3, 3, 4, 4},
{1, 1, 2, 3, 4, 4, 4}, {1, 1, 2, 4, 4, 4, 4}, {1, 1, 3, 3, 3, 3, 3},
{1, 1, 3, 3, 3, 3, 4}, {1, 1, 3, 3, 3, 4, 4}, {1, 1, 3, 3, 4, 4, 4},
{1, 1, 3, 4, 4, 4, 4}, {1, 2, 2, 2, 2, 2, 2}, {1, 2, 2, 2, 2, 2, 3},
{1, 2, 2, 2, 2, 2, 4}, {1, 2, 2, 2, 2, 3, 3}, {1, 2, 2, 2, 2, 3, 4},
{1, 2, 2, 2, 2, 4, 4}, {1, 2, 2, 2, 3, 3, 3}, {1, 2, 2, 2, 3, 3, 4},
{1, 2, 2, 2, 3, 4, 4}, {1, 2, 2, 2, 4, 4, 4}, {1, 2, 2, 3, 3, 3, 3},
{1, 2, 2, 3, 3, 3, 4}, {1, 2, 2, 3, 3, 4, 4}, {1, 2, 2, 3, 4, 4, 4},
{1, 2, 2, 4, 4, 4, 4}, {1, 2, 3, 3, 3, 3, 3}, {1, 2, 3, 3, 3, 3, 4},
```

```
{1, 2, 3, 3, 3, 4, 4}, {1, 2, 3, 3, 4, 4, 4}, {1, 2, 3, 4, 4, 4, 4},
{1, 3, 3, 3, 3, 3, 3}, {1, 3, 3, 3, 3, 3, 4}, {1, 3, 3, 3, 3, 4, 4},
{1, 3, 3, 3, 4, 4, 4}, {1, 3, 3, 4, 4, 4, 4}, {2, 2, 2, 2, 2, 2, 3},
{2, 2, 2, 2, 2, 3, 3}, {2, 2, 2, 2, 2, 3, 4}, {2, 2, 2, 2, 3, 3, 3},
{2, 2, 2, 2, 3, 3, 4}, {2, 2, 2, 2, 3, 4, 4}, {2, 2, 2, 3, 3, 3, 3},
{2, 2, 2, 3, 3, 3, 4}, {2, 2, 2, 3, 3, 4, 4}, {2, 2, 2, 3, 4, 4, 4},
{2, 2, 3, 3, 3, 3, 3}, {2, 2, 3, 3, 3, 3, 4}, {2, 2, 3, 3, 3, 4, 4},
{2, 2, 3, 3, 4, 4, 4}, {2, 2, 3, 4, 4, 4, 4}, {2, 3, 3, 3, 3, 3, 3},
{2, 3, 3, 3, 3, 3, 4}, {2, 3, 3, 3, 3, 4, 4}, {2, 3, 3, 3, 4, 4, 4},
{2, 3, 3, 4, 4, 4, 4}, {3, 3, 3, 3, 3, 3, 3}, {3, 3, 3, 3, 3, 3, 4},
{3, 3, 3, 3, 3, 4, 4}, {3, 3, 3, 3, 4, 4, 4}, {3, 3, 3, 4, 4, 4, 4}}

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```

The next code creates a list of all multi-sets of 7 elements from the set {1,2,3,4}

```
b = {1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4,
    4, 4, 4, 4, 4, 4};
sets = Subsets[b, {7}];
sets = Union[sets];
newsets = {};
For[i = 1, i <= Length[sets], i++,
 {value = Sort[Part[sets, i]],
  AppendTo[newsets, value]
  }
 ]
newsets = Union[newsets]
Length[newsets]

{{1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 2}, {1, 1, 1, 1, 1, 1, 3},
{1, 1, 1, 1, 1, 1, 4}, {1, 1, 1, 1, 1, 2, 2}, {1, 1, 1, 1, 1, 2, 3},
{1, 1, 1, 1, 1, 2, 4}, {1, 1, 1, 1, 1, 3, 3}, {1, 1, 1, 1, 1, 3, 4},
{1, 1, 1, 1, 1, 4, 4}, {1, 1, 1, 1, 2, 2, 2}, {1, 1, 1, 1, 2, 2, 3},
{1, 1, 1, 1, 2, 2, 4}, {1, 1, 1, 1, 2, 3, 3}, {1, 1, 1, 1, 2, 3, 4},
{1, 1, 1, 1, 2, 4, 4}, {1, 1, 1, 1, 3, 3, 3}, {1, 1, 1, 1, 3, 3, 4},
{1, 1, 1, 1, 3, 4, 4}, {1, 1, 1, 1, 4, 4, 4}, {1, 1, 1, 2, 2, 2, 2},
{1, 1, 1, 2, 2, 2, 3}, {1, 1, 1, 2, 2, 2, 4}, {1, 1, 1, 2, 2, 3, 3},
{1, 1, 1, 2, 2, 3, 4}, {1, 1, 1, 2, 2, 4, 4}, {1, 1, 1, 2, 3, 3, 3},
{1, 1, 1, 2, 3, 3, 4}, {1, 1, 1, 2, 3, 4, 4}, {1, 1, 1, 2, 4, 4, 4},
```

{1, 1, 1, 3, 3, 3, 3}, {1, 1, 1, 3, 3, 3, 4}, {1, 1, 1, 3, 3, 4, 4},
{1, 1, 1, 3, 4, 4, 4}, {1, 1, 1, 4, 4, 4, 4}, {1, 1, 2, 2, 2, 2, 2},
{1, 1, 2, 2, 2, 2, 3}, {1, 1, 2, 2, 2, 2, 4}, {1, 1, 2, 2, 2, 3, 3},
{1, 1, 2, 2, 2, 3, 4}, {1, 1, 2, 2, 2, 4, 4}, {1, 1, 2, 2, 3, 3, 3},
{1, 1, 2, 2, 3, 3, 4}, {1, 1, 2, 2, 3, 4, 4}, {1, 1, 2, 2, 4, 4, 4},
{1, 1, 2, 3, 3, 3, 3}, {1, 1, 2, 3, 3, 3, 4}, {1, 1, 2, 3, 3, 4, 4},
{1, 1, 2, 3, 4, 4, 4}, {1, 1, 2, 4, 4, 4, 4}, {1, 1, 3, 3, 3, 3, 3},
{1, 1, 3, 3, 3, 3, 4}, {1, 1, 3, 3, 3, 4, 4}, {1, 1, 3, 3, 4, 4, 4},
{1, 1, 3, 4, 4, 4, 4}, {1, 1, 4, 4, 4, 4, 4}, {1, 2, 2, 2, 2, 2, 2},
{1, 2, 2, 2, 2, 2, 3}, {1, 2, 2, 2, 2, 2, 4}, {1, 2, 2, 2, 2, 3, 3},
{1, 2, 2, 2, 2, 3, 4}, {1, 2, 2, 2, 2, 4, 4}, {1, 2, 2, 2, 3, 3, 3},
{1, 2, 2, 2, 3, 3, 4}, {1, 2, 2, 2, 3, 4, 4}, {1, 2, 2, 2, 4, 4, 4},
{1, 2, 2, 3, 3, 3, 3}, {1, 2, 2, 3, 3, 3, 4}, {1, 2, 2, 3, 3, 4, 4},
{1, 2, 2, 3, 4, 4, 4}, {1, 2, 2, 4, 4, 4, 4}, {1, 2, 3, 3, 3, 3, 3},
{1, 2, 3, 3, 3, 3, 4}, {1, 2, 3, 3, 3, 4, 4}, {1, 2, 3, 3, 4, 4, 4},
{1, 2, 3, 4, 4, 4, 4}, {1, 2, 4, 4, 4, 4, 4}, {1, 3, 3, 3, 3, 3, 3},
{1, 3, 3, 3, 3, 3, 4}, {1, 3, 3, 3, 3, 4, 4}, {1, 3, 3, 3, 4, 4, 4},
{1, 3, 3, 4, 4, 4, 4}, {1, 3, 4, 4, 4, 4, 4}, {1, 4, 4, 4, 4, 4, 4},
{2, 2, 2, 2, 2, 2, 2}, {2, 2, 2, 2, 2, 2, 3}, {2, 2, 2, 2, 2, 2, 4},
{2, 2, 2, 2, 2, 3, 3}, {2, 2, 2, 2, 2, 3, 4}, {2, 2, 2, 2, 2, 4, 4},
{2, 2, 2, 2, 3, 3, 3}, {2, 2, 2, 2, 3, 3, 4}, {2, 2, 2, 2, 3, 4, 4},
{2, 2, 2, 2, 4, 4, 4}, {2, 2, 2, 3, 3, 3, 3}, {2, 2, 2, 3, 3, 3, 4},
{2, 2, 2, 3, 3, 4, 4}, {2, 2, 2, 3, 4, 4, 4}, {2, 2, 2, 4, 4, 4, 4},
{2, 2, 3, 3, 3, 3, 3}, {2, 2, 3, 3, 3, 3, 4}, {2, 2, 3, 3, 3, 4, 4},
{2, 2, 3, 3, 4, 4, 4}, {2, 2, 3, 4, 4, 4, 4}, {2, 2, 4, 4, 4, 4, 4},
{2, 3, 3, 3, 3, 3, 3}, {2, 3, 3, 3, 3, 3, 4}, {2, 3, 3, 3, 3, 4, 4},
{2, 3, 3, 3, 4, 4, 4}, {2, 3, 3, 4, 4, 4, 4}, {2, 3, 4, 4, 4, 4, 4},
{2, 4, 4, 4, 4, 4, 4}, {3, 3, 3, 3, 3, 3, 3}, {3, 3, 3, 3, 3, 3, 4},
{3, 3, 3, 3, 3, 4, 4}, {3, 3, 3, 3, 4, 4, 4}, {3, 3, 3, 4, 4, 4, 4},
{3, 3, 4, 4, 4, 4, 4}, {3, 4, 4, 4, 4, 4, 4}, {4, 4, 4, 4,4, 4, 4}}

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# G Complete Solution of Marco Buratti's Problem for $s = 9$

We first include a table that includes every possible set $S$ of lengths in a Hamiltonian path along with a Hamiltonian path in $K_9$ whose edge lengths correspond to the values in the multi-set $S$ when a Hamiltonian path exists.

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{1,1,1,1,1,1,1,1\}$ | $\{0,1,2,3,4,5,6,7,8\}$ | $\{1,1,1,1,1,1,1,2\}$ | $\{0,1,2,3,4,5,6,8,7\}$ |
| $\{1,1,1,1,1,1,1,3\}$ | $\{0,1,2,3,4,5,8,7,6\}$ | $\{1,1,1,1,1,1,1,4\}$ | $\{0,1,2,3,4,8,7,6,5\}$ |
| $\{1,1,1,1,1,1,2,2\}$ | $\{0,1,2,3,5,4,6,7,8\}$ | $\{1,1,1,1,1,1,2,3\}$ | $\{0,1,2,8,7,6,5,3,4\}$ |
| $\{1,1,1,1,1,1,2,4\}$ | $\{0,1,8,7,2,3,4,5,6\}$ | $\{1,1,1,1,1,1,3,3\}$ | $\{0,3,2,1,4,5,6,7,8\}$ |
| $\{1,1,1,1,1,1,3,4\}$ | $\{0,8,3,2,1,4,5,6,7\}$ | $\{1,1,1,1,1,1,4,4\}$ | $\{0,8,3,4,5,6,7,2,1\}$ |
| $\{1,1,1,1,1,2,2,2\}$ | $\{0,8,6,7,5,4,3,1,2\}$ | $\{1,1,1,1,1,2,2,3\}$ | $\{0,8,7,1,2,3,5,4,6\}$ |
| $\{1,1,1,1,1,2,2,4\}$ | $\{0,8,7,2,1,3,4,6,5\}$ | $\{1,1,1,1,1,2,3,3\}$ | $\{0,8,7,4,1,2,3,5,6\}$ |
| $\{1,1,1,1,1,2,3,4\}$ | $\{0,8,7,4,3,1,2,6,5\}$ | $\{1,1,1,1,1,2,4,4\}$ | $\{0,8,7,5,4,3,2,6,1\}$ |
| $\{1,1,1,1,1,3,3,3\}$ | $\{0,8,7,6,3,2,5,4,1\}$ | $\{1,1,1,1,1,3,3,4\}$ | $\{0,1,2,3,8,5,4,7,6\}$ |
| $\{1,1,1,1,1,3,4,4\}$ | $\{0,1,2,5,6,7,3,4,8\}$ | $\{1,1,1,1,1,4,4,4\}$ | $\{0,1,2,6,5,4,8,7,3\}$ |
| $\{1,1,1,1,2,2,2,2\}$ | $\{0,1,3,2,4,5,7,6,8\}$ | $\{1,1,1,1,2,2,2,3\}$ | $\{0,1,3,2,4,5,8,6,7\}$ |
| $\{1,1,1,1,2,2,2,4\}$ | $\{0,1,3,2,4,8,7,5,6\}$ | $\{1,1,1,1,2,2,3,3\}$ | $\{0,1,3,2,5,4,7,6,8\}$ |
| $\{1,1,1,1,2,2,3,4\}$ | $\{0,1,3,2,5,4,8,6,7\}$ | $\{1,1,1,1,2,2,4,4\}$ | $\{0,1,3,2,6,5,7,8,4\}$ |
| $\{1,1,1,1,2,3,3,3\}$ | $\{0,1,3,2,8,5,4,7,6\}$ | $\{1,1,1,1,2,3,3,4\}$ | $\{0,1,3,4,5,8,2,6,7\}$ |
| $\{1,1,1,1,2,3,4,4\}$ | $\{0,1,3,4,5,8,7,2,6\}$ | $\{1,1,1,1,2,4,4,4\}$ | $\{0,1,3,4,8,7,2,6,5\}$ |
| $\{1,1,1,1,3,3,3,3\}$ | $\{0,1,4,3,6,5,2,8,7\}$ | $\{1,1,1,1,3,3,3,4\}$ | $\{0,1,4,3,6,5,8,7,2\}$ |
| $\{1,1,1,1,3,3,4,4\}$ | $\{0,1,4,3,7,8,2,6,5\}$ | $\{1,1,1,1,3,4,4,4\}$ | $\{0,1,4,3,8,7,2,6,5\}$ |
| $\{1,1,1,1,4,4,4,4\}$ | $\{0,1,5,4,3,8,7,2,6\}$ | $\{1,1,1,2,2,2,2,2\}$ | $\{0,1,8,6,7,5,3,2,4\}$ |
| $\{1,1,1,2,2,2,2,3\}$ | $\{0,1,8,7,6,4,2,5,3\}$ | $\{1,1,1,2,2,2,2,4\}$ | $\{0,2,1,3,4,8,6,5,7\}$ |
| $\{1,1,1,2,2,2,3,3\}$ | $\{0,2,1,3,6,4,5,8,7\}$ | $\{1,1,1,2,2,2,3,4\}$ | $\{0,2,1,3,6,5,7,8,4\}$ |
| $\{1,1,1,2,2,2,4,4\}$ | $\{0,2,1,3,7,6,8,4,5\}$ | $\{1,1,1,2,2,3,3,3\}$ | $\{0,2,1,4,3,6,7,5,8\}$ |
| $\{1,1,1,2,2,3,3,4\}$ | $\{0,2,1,4,3,7,6,8,5\}$ | $\{1,1,1,2,2,3,4,4\}$ | $\{0,2,1,4,5,6,8,3,7\}$ |
| $\{1,1,1,2,2,4,4,4\}$ | $\{0,2,1,5,4,6,7,3,8\}$ | $\{1,1,1,2,3,3,3,3\}$ | $\{0,2,1,7,4,3,6,5,8\}$ |
| $\{1,1,1,2,3,3,3,4\}$ | $\{0,2,1,7,4,3,8,5,6\}$ | $\{1,1,1,2,3,3,4,4\}$ | $\{0,2,1,7,6,3,8,4,5\}$ |
| $\{1,1,1,2,3,4,4,4\}$ | $\{0,2,3,4,8,5,1,6,7\}$ | $\{1,1,1,2,4,4,4,4\}$ | $\{0,2,3,7,6,1,5,4,8\}$ |
| $\{1,1,1,3,3,3,3,3\}$ | $\{0,3,2,8,5,4,1,7,6\}$ | $\{1,1,1,3,3,3,3,4\}$ | $\{0,3,2,8,5,4,7,6,1\}$ |

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{1,1,1,3,3,3,4,4\}$ | $\{0,3,2,8,7,4,5,1,6\}$ | $\{1,1,1,3,3,4,4,4\}$ | $\{0,3,4,1,5,6,2,7,8\}$ |
| $\{1,1,1,3,4,4,4,4\}$ | $\{0,3,4,5,1,6,2,7,8\}$ | $\{1,1,1,4,4,4,4,4\}$ | $\{0,4,3,8,7,2,6,5,1\}$ |
| $\{1,1,2,2,2,2,2,2\}$ | $\{0,7,5,3,2,1,8,6,4\}$ | $\{1,1,2,2,2,2,2,3\}$ | $\{0,7,5,3,4,6,8,2,1\}$ |
| $\{1,1,2,2,2,2,2,4\}$ | $\{0,7,5,4,2,1,3,8,6\}$ | $\{1,1,2,2,2,2,3,3\}$ | $\{0,7,5,4,6,3,1,2,8\}$ |
| $\{1,1,2,2,2,2,3,4\}$ | $\{0,7,5,4,6,3,8,1,2\}$ | $\{1,1,2,2,2,2,4,4\}$ | $\{0,7,5,4,8,1,3,2,6\}$ |
| $\{1,1,2,2,2,3,3,3\}$ | $\{0,7,5,6,3,2,8,1,4\}$ | $\{1,1,2,2,2,3,3,4\}$ | $\{0,7,5,6,4,1,2,8,3\}$ |
| $\{1,1,2,2,2,3,4,4\}$ | $\{0,7,5,8,1,6,2,3,4\}$ | $\{1,1,2,2,2,4,4,4\}$ | $\{0,7,6,1,2,4,8,3,5\}$ |
| $\{1,1,2,2,3,3,3,3\}$ | $\{0,7,6,3,1,4,5,2,8\}$ | $\{1,1,2,2,3,3,3,4\}$ | $\{0,7,6,3,2,4,1,5,8\}$ |
| $\{1,1,2,2,3,3,4,4\}$ | $\{0,7,6,3,2,4,8,5,1\}$ | $\{1,1,2,2,3,4,4,4\}$ | $\{0,7,6,3,8,4,2,1,5\}$ |
| $\{1,1,2,2,4,4,4,4\}$ | $\{0,7,8,3,4,2,6,1,5\}$ | $\{1,1,2,3,3,3,3,3\}$ | $\{0,8,2,5,3,6,7,1,4\}$ |
| $\{1,1,2,3,3,3,3,4\}$ | $\{0,8,2,5,4,6,3,7,1\}$ | $\{1,1,2,3,3,3,4,4\}$ | $\{0,8,2,5,4,7,3,1,6\}$ |
| $\{1,1,2,3,3,4,4,4\}$ | $\{0,8,2,6,1,3,7,4,5\}$ | $\{1,1,2,3,4,4,4,4\}$ | $\{0,8,2,6,1,5,7,3,4\}$ |
| $\{1,1,2,4,4,4,4,4\}$ | $\{0,8,3,7,2,4,5,1,6\}$ | $\{1,1,3,3,3,3,3,3\}$ | $\{0,3,6,5,2,8,7,1,4\}$ |
| $\{1,1,3,3,3,3,3,4\}$ | $\{0,3,6,5,4,1,7,2,8\}$ | $\{1,1,3,3,3,3,4,4\}$ | $\{0,3,6,5,4,8,2,7,1\}$ |
| $\{1,1,3,3,3,4,4,4\}$ | $\{0,3,6,7,2,1,5,8,4\}$ | $\{1,1,3,3,4,4,4,4\}$ | $\{0,3,7,2,1,6,5,8,4\}$ |
| $\{1,1,3,4,4,4,4,4\}$ | $\{0,3,7,8,4,5,1,6,2\}$ | $\{1,1,4,4,4,4,4,4\}$ | $\{0,4,3,8,7,2,6,1,5\}$ |
| $\{1,2,2,2,2,2,2,2\}$ | $\{0,7,5,3,1,2,4,6,8\}$ | $\{1,2,2,2,2,2,2,3\}$ | $\{0,7,5,3,1,2,8,6,4\}$ |
| $\{1,2,2,2,2,2,2,4\}$ | $\{0,7,5,3,2,4,6,1,8\}$ | $\{1,2,2,2,2,2,3,3\}$ | $\{0,7,5,3,2,8,1,4,6\}$ |
| $\{1,2,2,2,2,2,3,4\}$ | $\{0,7,5,3,2,8,1,6,4\}$ | $\{1,2,2,2,2,2,4,4\}$ | $\{0,7,5,3,4,2,6,1,8\}$ |
| $\{1,2,2,2,2,3,3,3\}$ | $\{0,7,5,3,6,4,1,2,8\}$ | $\{1,2,2,2,2,3,3,4\}$ | $\{0,7,5,3,6,4,8,2,1\}$ |
| $\{1,2,2,2,2,3,4,4\}$ | $\{0,7,5,3,8,2,1,6,4\}$ | $\{1,2,2,2,2,4,4,4\}$ | $\{0,7,5,3,8,4,2,1,6\}$ |
| $\{1,2,2,2,3,3,3,3\}$ | $\{0,7,5,8,2,1,4,6,3\}$ | $\{1,2,2,2,3,3,3,4\}$ | $\{0,7,5,8,2,4,3,6,1\}$ |
| $\{1,2,2,2,3,3,4,4\}$ | $\{0,7,5,8,2,6,1,3,4\}$ | $\{1,2,2,2,3,4,4,4\}$ | $\{0,7,5,8,3,4,2,6,1\}$ |
| $\{1,2,2,2,4,4,4,4\}$ | $\{0,7,6,1,5,3,8,4,2\}$ | $\{1,2,2,3,3,3,3,3\}$ | $\{0,8,2,5,3,6,4,1,7\}$ |
| $\{1,2,2,3,3,3,3,4\}$ | $\{0,8,2,5,3,7,1,4,6\}$ | $\{1,2,2,3,3,3,4,4\}$ | $\{0,8,2,5,3,7,1,6,4\}$ |
| $\{1,2,2,3,3,4,4,4\}$ | $\{0,8,2,6,4,1,5,3,7\}$ | $\{1,2,2,3,4,4,4,4\}$ | $\{0,8,2,7,3,5,1,6,4\}$ |
| $\{1,2,2,4,4,4,4,4\}$ | $\{0,8,3,7,2,4,6,1,5\}$ | $\{1,2,3,3,3,3,3,3\}$ | $\{0,3,6,4,1,7,8,2,5\}$ |
| $\{1,2,3,3,3,3,3,4\}$ | $\{0,3,6,4,5,8,2,7,1\}$ | $\{1,2,3,3,3,3,4,4\}$ | $\{0,3,6,4,7,2,1,5,8\}$ |
| $\{1,2,3,3,3,4,4,4\}$ | $\{0,3,6,4,8,5,1,2,7\}$ | $\{1,2,3,3,4,4,4,4\}$ | $\{0,3,7,1,5,4,8,6,2\}$ |
| $\{1,2,3,4,4,4,4,4\}$ | $\{0,3,7,2,6,1,8,4,5\}$ | $\{1,2,4,4,4,4,4,4\}$ | $\{0,4,5,1,6,8,3,7,2\}$ |
| $\{1,3,3,3,3,3,3,3\}$ | $DNE$ | $\{1,3,3,3,3,3,3,4\}$ | $\{0,6,3,2,5,8,4,1,7\}$ |
| $\{1,3,3,3,3,3,4,4\}$ | $\{0,6,3,2,7,1,4,8,5\}$ | $\{1,3,3,3,3,4,4,4\}$ | $\{0,6,3,2,7,1,5,8,4\}$ |
| $\{1,3,3,3,4,4,4,4\}$ | $\{0,6,3,7,2,1,5,8,4\}$ | $\{1,3,3,4,4,4,4,4\}$ | $\{0,6,5,1,4,8,3,7,2\}$ |
| $\{1,3,4,4,4,4,4,4\}$ | $\{0,3,7,2,6,1,5,4,8\}$ | $\{1,4,4,4,4,4,4,4\}$ | $\{0,4,5,1,6,2,7,3,8\}$ |
| $\{2,2,2,2,2,2,2,2\}$ | $\{0,7,5,3,1,8,6,4,2\}$ | $\{2,2,2,2,2,2,2,3\}$ | $\{0,2,4,6,8,1,7,5,3\}$ |
| $\{2,2,2,2,2,2,2,4\}$ | $\{0,2,7,5,3,1,8,6,4\}$ | $\{2,2,2,2,2,2,3,3\}$ | $\{0,2,8,1,3,5,7,4,6\}$ |
| $\{2,2,2,2,2,2,3,4\}$ | $\{0,3,1,8,6,4,2,7,5\}$ | $\{2,2,2,2,2,2,4,4\}$ | $\{0,4,2,6,8,1,3,5,7\}$ |

| S | $HamiltonianPath$ | S | $HamiltonianPath$ |
|---|---|---|---|
| $\{2,2,2,2,2,3,3,3\}$ | $\{0,6,4,2,8,1,7,5,3\}$ | $\{2,2,2,2,2,3,3,4\}$ | $\{0,6,8,1,3,7,5,2,4\}$ |
| $\{2,2,2,2,2,3,4,4\}$ | $\{0,6,8,4,2,7,5,3,1\}$ | $\{2,2,2,2,2,4,4,4\}$ | $\{0,7,2,4,6,1,8,3,5\}$ |
| $\{2,2,2,2,3,3,3,3\}$ | $\{0,7,4,1,3,5,2,8,6\}$ | $\{2,2,2,2,3,3,3,4\}$ | $\{0,7,4,1,3,5,8,6,2\}$ |
| $\{2,2,2,2,3,3,4,4\}$ | $\{0,7,4,2,5,1,3,8,6\}$ | $\{2,2,2,2,3,4,4,4\}$ | $\{0,7,4,2,6,1,8,3,5\}$ |
| $\{2,2,2,2,4,4,4,4\}$ | $\{0,7,5,1,3,8,4,2,6\}$ | $\{2,2,2,3,3,3,3,3\}$ | $\{0,2,5,8,1,7,4,6,3\}$ |
| $\{2,2,2,3,3,3,3,4\}$ | $\{0,2,5,8,3,1,7,4,6\}$ | $\{2,2,2,3,3,3,4,4\}$ | $\{0,2,5,8,6,1,3,7,4\}$ |
| $\{2,2,2,3,3,4,4,4\}$ | $\{0,2,6,1,3,5,8,4,7\}$ | $\{2,2,2,3,4,4,4,4\}$ | $\{0,2,6,1,3,7,5,8,4\}$ |
| $\{2,2,2,4,4,4,4,4\}$ | $\{0,2,6,4,8,3,7,5,1\}$ | $\{2,2,3,3,3,3,3,3\}$ | $\{0,3,6,4,1,7,5,2,8\}$ |
| $\{2,2,3,3,3,3,3,4\}$ | $\{0,3,6,4,1,8,5,2,7\}$ | $\{2,2,3,3,3,3,4,4\}$ | $\{0,3,6,4,2,7,1,5,8\}$ |
| $\{2,2,3,3,3,4,4,4\}$ | $\{0,3,6,4,8,1,5,2,7\}$ | $\{2,2,3,3,4,4,4,4\}$ | $\{0,3,7,2,4,6,1,5,8\}$ |
| $\{2,2,3,4,4,4,4,4\}$ | $\{0,3,7,2,4,8,6,1,5\}$ | $\{2,2,4,4,4,4,4,4\}$ | $\{0,4,2,6,1,5,7,3,8\}$ |
| $\{2,3,3,3,3,3,3,3\}$ | $DNE$ | $\{2,3,3,3,3,3,3,4\}$ | $\{0,6,3,1,4,7,2,5,8\}$ |
| $\{2,3,3,3,3,3,4,4\}$ | $\{0,6,3,1,4,8,5,2,7\}$ | $\{2,3,3,3,3,4,4,4\}$ | $\{0,6,3,1,5,2,7,4,8\}$ |
| $\{2,3,3,3,4,4,4,4\}$ | $\{0,6,3,7,2,4,8,5,1\}$ | $\{2,3,3,4,4,4,4,4\}$ | $\{0,6,4,8,3,7,2,5,1\}$ |
| $\{2,3,4,4,4,4,4,4\}$ | $\{0,4,2,6,1,5,8,3,7\}$ | $\{2,4,4,4,4,4,4,4\}$ | $\{0,4,8,3,5,1,6,2,7\}$ |
| $\{3,3,3,3,3,3,3,3\}$ | $DNE$ | $\{3,3,3,3,3,3,3,4\}$ | $DNE$ |
| $\{3,3,3,3,3,3,4,4\}$ | $\{0,6,3,7,1,4,8,2,5\}$ | $\{3,3,3,3,3,4,4,4\}$ | $\{0,6,3,7,1,5,2,8,4\}$ |
| $\{3,3,3,3,4,4,4,4\}$ | $\{0,6,3,7,2,5,1,4,8\}$ | $\{3,3,3,4,4,4,4,4\}$ | $\{0,3,7,2,6,1,4,8,5\}$ |
| $\{3,3,4,4,4,4,4,4\}$ | $\{0,3,7,2,6,1,5,8,4\}$ | $\{3,4,4,4,4,4,4,4\}$ | $\{0,4,8,3,7,2,5,1,6\}$ |
| $\{4,4,4,4,4,4,4,4\}$ | $\{0,4,8,3,7,2,6,1,5\}$ | | |

Included next is the *Mathematica* code used to determine which sets $S$ for $s = 9$ are realizable.

```
First, create a list of all permutations on 9 elements.

a = {1, 2, 3, 4, 5, 6, 7, 8, 9};
perms = Permutations[a];
Length[perms]

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Calculate the differences between consecutive elements in each permutation.

diff = {};
For[j = 1, j <= Length[perms], j++,
 {diffs = {0, 0, 0, 0, 0, 0, 0, 0},
  perm = Part[perms, j],
  For[i = 1, i <= 8, i++,
   {value = Part[perm, i + 1] - Part[perm, i],
    Part[diffs, i] = value
    }
   ],
  AppendTo[diff, diffs]
  }
 ]


Relabel the elements of the difference sets to be between 1 and 4 and remove repeated elements.

diff = diff /. -1 -> 1;
diff = diff /. -2 -> 2;
diff = diff /. -3 -> 3;
diff = diff /. -4 -> 4;
diff = diff /. -5 -> 5;
diff = diff /. -6 -> 6;
diff = diff /. -7 -> 7;
diff = diff /. -8 -> 8;
diff = diff /. 5 -> 4;
diff = diff /. 6 -> 3;
```

```
diff = diff /. 7 -> 2;
diff = diff /. 8 -> 1;

diff = Union[diff];
newlist = {};
For[k = 1, k <= Length[diff], k++,
 {part = Sort[Part[diff, k]],
  AppendTo[newlist, part]
  }
 ]
newlist = Union[newlist]
Length[newlist]
```

```
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 2}, {1, 1, 1, 1, 1, 1, 1, 3},
{1, 1, 1, 1, 1, 1, 1, 4}, {1, 1, 1, 1, 1, 1, 2, 2}, {1, 1, 1, 1, 1, 1, 2, 3},
{1, 1, 1, 1, 1, 1, 2, 4}, {1, 1, 1, 1, 1, 1, 3, 3}, {1, 1, 1, 1, 1, 1, 3, 4},
{1, 1, 1, 1, 1, 1, 4, 4}, {1, 1, 1, 1, 1, 2, 2, 2}, {1, 1, 1, 1, 1, 2, 2, 3},
{1, 1, 1, 1, 1, 2, 2, 4}, {1, 1, 1, 1, 1, 2, 3, 3}, {1, 1, 1, 1, 1, 2, 3, 4},
{1, 1, 1, 1, 1, 2, 4, 4}, {1, 1, 1, 1, 1, 3, 3, 3}, {1, 1, 1, 1, 1, 3, 3, 4},
{1, 1, 1, 1, 1, 3, 4, 4}, {1, 1, 1, 1, 1, 4, 4, 4}, {1, 1, 1, 1, 2, 2, 2, 2},
{1, 1, 1, 1, 2, 2, 2, 3}, {1, 1, 1, 1, 2, 2, 2, 4}, {1, 1, 1, 1, 2, 2, 3, 3},
{1, 1, 1, 1, 2, 2, 3, 4}, {1, 1, 1, 1, 2, 2, 4, 4}, {1, 1, 1, 1, 2, 3, 3, 3},
{1, 1, 1, 1, 2, 3, 3, 4}, {1, 1, 1, 1, 2, 3, 4, 4}, {1, 1, 1, 1, 2, 4, 4, 4},
{1, 1, 1, 1, 3, 3, 3, 3}, {1, 1, 1, 1, 3, 3, 3, 4}, {1, 1, 1, 1, 3, 3, 4, 4},
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{1, 1, 2, 2, 2, 2, 3, 4}, {1, 1, 2, 2, 2, 2, 4, 4}, {1, 1, 2, 2, 2, 3, 3, 3},
{1, 1, 2, 2, 2, 3, 3, 4}, {1, 1, 2, 2, 2, 3, 4, 4}, {1, 1, 2, 2, 2, 4, 4, 4},
{1, 1, 2, 2, 3, 3, 3, 3}, {1, 1, 2, 2, 3, 3, 3, 4}, {1, 1, 2, 2, 3, 3, 4, 4},
{1, 1, 2, 2, 3, 4, 4, 4}, {1, 1, 2, 2, 4, 4, 4, 4}, {1, 1, 2, 3, 3, 3, 3, 3},
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```

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{1, 2, 2, 2, 2, 2, 3, 3}, {1, 2, 2, 2, 2, 2, 3, 4}, {1, 2, 2, 2, 2, 2, 4, 4},
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{2, 2, 2, 3, 4, 4, 4, 4}, {2, 2, 2, 4, 4, 4, 4, 4}, {2, 2, 3, 3, 3, 3, 3, 3},
{2, 2, 3, 3, 3, 3, 3, 4}, {2, 2, 3, 3, 3, 3, 4, 4}, {2, 2, 3, 3, 3, 4, 4, 4},
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{2, 3, 3, 3, 4, 4, 4, 4}, {2, 3, 3, 4, 4, 4, 4, 4}, {2, 3, 4, 4, 4, 4, 4, 4},
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{3, 3, 3, 3, 4, 4, 4, 4}, {3, 3, 3, 4, 4, 4, 4, 4}, {3, 3, 4, 4, 4, 4, 4, 4},
{3, 4, 4, 4, 4, 4, 4, 4}, {4, 4, 4, 4, 4, 4, 4, 4}}

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In[106]:=
b = {1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3,
    3, 3, 4, 4, 4, 4, 4, 4, 4, 4};
sets = Subsets[b, {8}];
sets = Union[sets];
newsets = {};
For[i = 1, i <= Length[sets], i++,
 {value = Sort[Part[sets, i]],
  AppendTo[newsets, value]
  }
```

```
 ]
newsets = Union[newsets]
Length[newsets]

Out[111]=
{{1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 2}, {1, 1,1, 1, 1, 1, 1, 3},
{1, 1, 1, 1, 1, 1, 1, 4}, {1, 1, 1, 1, 1, 1, 2, 2}, {1, 1, 1, 1, 1, 1, 2, 3},
{1, 1, 1, 1, 1, 1, 2, 4}, {1, 1, 1, 1, 1, 1, 3, 3}, {1, 1, 1, 1, 1, 1, 3, 4},
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{1, 1, 1, 3, 3, 3, 3, 4}, {1, 1, 1, 3, 3, 3, 4, 4}, {1, 1, 1, 3, 3, 4, 4, 4},
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