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Popescu's Conjecture in Multiquadratic Extensions

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POPESCU'S CONJECTURE IN MULTIQUADRATIC EXTENSIONS

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Abstract

Stark's Conjectures were formulated in the late 1970s and early 1980s. The most general version predicts that the leading coefficient of the Maclaurin series of an Artin L-function should be the product of an algebraic number and a regulator made up of character values and logarithms of absolute values of units. When known, Stark's conjecture provides a factorization of the analytic class number formula of Dirichlet. Stark succeeded in formulating a "refined abelian" version of his conjecture when the L-function in question has a first order zero and is associated with an abelian extension of number fields.

In the spirit of Stark, Rubin and Popescu formulated analogous "refined abelian" conjectures for Artin L-Functions which vanish to arbitrary order r at $s = 0$. These conjectures are identical to Stark's own refined abelian conjecture when restricted to order of vanishing $r = 1$. We introduce Popescu's Conjecture $C(L/F, S, r)$. We prove Popescu's Conjecture for multiquadratic extensions when the set of primes S of the base field is minimal given minor restrictions on the S-class group of the base field. This extends the results of Sands to the case where $\#S = r + 1$. We present three infinite families of settings where our methods allow us to verify Popescu's conjecture. We formulate a conjecture that predicts when a fundamental unit of a real quadratic field must become a square in a multiquadratic extension.

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Chapter 1

Introduction

Popescu's conjecture [16] represents a generalization of Stark's refined abelian conjecture [22]. When K/F is an abelian extension of number fields and S is finite set of primes of F containing all infinite and ramified primes as well as at laest one prime v that splits completely in K/F, the latter conjecture posits the existence of a distinguished S-unit, $\epsilon_{K/F,S}$, of K. This element is known as the "Stark unit" for L/K and is conjectured to satisfy remarkable arithmetic properties. The element $\epsilon_{K/F,S}$ evaluates the first derivatives at $s = 0$ of all Artin L-functions $L_{K/F,S}(s, \chi)$ associated to the extension K/F. Stark's refined abelian conjecture holds trivially for a large class of extensions. If all Artin L-functions associated to K/F vanish to order larger than one at $s = 0$ then the conjecture holds with $\epsilon_{K/F,S} = 1$. This occurs, for example, when the base field $F \neq \mathbb{Q}$ is a totally real field and K is also totally real.

Popescu's conjecture, $C(K/F, S, r)$, extends Stark's refined abelian conjecture to L-functions whose derivatives at $s = 0$ vanish to arbitrary order r. This order of vanishing is accomplished by requiring that S contain r primes that split completely in K/F and no fewer than $r + 1$ primes in total. Popescu's conjecture also specifies an L-function evaluator $\epsilon_{K/F,S}$ which will no longer be a single S-unit. The conjecture predicts that

$\epsilon_{K/F,S}$ will be an element of a certain sublattice of a vector space generated by the r-th exterior power of the S-units of K. Upon applying a regulator map to $\epsilon_{K/F,S}$, one is able to evaluate the r-th derivatives of all Artin L-functions for K/F. Popescu has proven that this conjecture satisfies a base change property that allows one to ascertain its truth in some cases provided that one knows that Stark's refined abelian conjecture holds for a related extension.

Popescu's conjecture is is known to hold, for example, when K/F is a quadratic extension [18]. It is known to hold when $r = 1$ and Stark's refined abelian conjecture holds as the conjectures coincide. It is also known when one may apply the base-change property to a setting where Stark's refined abelian conjecture is known.

Sands [20] has identified the evaluator $\epsilon_{L/F}$ in the setting where the extension is multiquadratic. In this setting, $\epsilon_{L/F}$ is a sum of the evaluators associated to the relative quadratic subextensions of L/F. He proved Popescu's conjecture for such extensions given mild restrictions on the class group $Cl_{F,S}$ when $\#S > r + 1$. He also proposes a weaker conjecture $C'(L/F, S, r)$ and proves it for all such multiquadratic extensions. The restriction $\#S > r + 1$ ensures that the Dedekind zeta function vanishes to order $r+1$ at $s = 0$. The goal of our research is to prove the conjecture provided $\#S = r + 1$.

In Chapter 2, we discuss the algebraic and analytic objects that are used to formulate the conjectures and discuss the history of the conjectures. In Chapter 3 we explicitly write out the evaluator element $\epsilon_{L/F,S}$ where L/F is multiquadratic for the special case $\#S = r+1$. We prove a lemma (3.2.1) that enables us to write $\epsilon_{L/F,S}$ in a more compact form. We are then able to prove (Corollary 3.3.4) Popescu's conjecture under the same class group restrictions prescribed by Sands provided $\#S = r + 1$. We prove conjecture $C'(L/F, S, r)$ when $\#S = r + 1$ provided the class number $h_{F,S}$ is divisible by a sufficiently large power of 2 (Corollary 3.1.10). We prove a lemma (3.3.5) that establishes a lower bound for the power of 2 dividing $h_{F,S}$ and a lemma (3.4.1) which gives a sufficient condition to relax the power of 2 required to divide $h_{F,S}$ by one.

In Chapter 4 we present three infinite families of settings $(L/F, S, r)$ in which the results of Chapter 3 allow us to prove the full conjecture. In each family there is a maximal extension $\mathcal{L}/F$ where we are unable to prove the full conjecture. In Theorems 4.1.2 and 4.1.3 we are able to prove the weaker conjecture $C'(L/F, S, r)$ for the maximal quadratic extension in the corresponding families. In Theorem 4.1.1 we state a sufficient condition to prove the weak conjecture for the maximal extension and develop a routine to computationally verify when this condition is satisfied.

Chapter 2

The Elements of Popescu's Conjecture

2.1 Algebraic Number Fields

Let $\mathbb{Z}$ denote the ring of all integers $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$. Let $\mathbb{Q}$ denote the field of all rational numbers, that is the set of all reasonable (i.e. those without zero in the denominator) quotients of integers in their lowest terms. The rational numbers are an example of an algebraic number field.

Definition 2.1.1. An **algebraic number field** is a field that contains $\mathbb{Q}$ and is finite dimensional when considered as a vector space over $\mathbb{Q}$.

For the duration of this dissertation, all fields discussed will be algebraic number fields. Typically one obtains an algebraic number field by "tossing in" something new into $\mathbb{Q}$. For instance we may define the field F to be the smallest number field that contains the real number $\sqrt{165}$. Then $F = \mathbb{Q}(\sqrt{165})$ has dimension two as a vector space over $\mathbb{Q}$. The elements 1 and $\sqrt{165}$ yield a $\mathbb{Q}$-basis for F. The dimension of a number field over $\mathbb{Q}$ is called the *degree* of a number field.

A number field F may be viewed as a subfield of $\mathbb{C}$ and one may consider the various embeddings of F inside $\mathbb{C}$. The field $F = \mathbb{Q}(\sqrt{165})$, for example permits two distinct embeddings. The first, 1_F is the identity map. The second, τ can be defined by specifying that each rational number is mapped to itself while $\tau : \sqrt{165} \longmapsto -\sqrt{165}$. Both embeddings actually embed F in the set of real numbers $\mathbb{R}$. Such embeddings are called *real embeddings* while all others are called *complex*. If r_1 is the number of real embeddings of a number field F, r_2 is the number of complex embeddings, and n is the degree of F then one has the equality $n = r_1 + 2r_2$ [13, page 30].

The *ring of integers* $\mathcal{O}_F$ of a number field F is made up of all elements $\alpha \in F$ that satisfy a polynomial equation of the form $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n = 0$ where each coefficient $a_i \in \mathbb{Z}$. The integers $\mathbb{Z}$ form the ring of integers of $\mathbb{Q}$. This is why $\mathbb{Z}$ is often referred to as the set of rational integers. If $F = \mathbb{Q}(\sqrt{165})$ then its ring of integers is $\mathcal{O}_F = \mathbb{Z}\left[\frac{1+\sqrt{165}}{2}\right]$. The elements 1 and $\frac{1+\sqrt{165}}{2}$ give a $\mathbb{Z}$-basis for $\mathcal{O}_F$ and a $\mathbb{Q}$-basis for F. A basis with these two properties is called an *integral basis* for $\mathcal{O}_F$. An integral basis is used to compute the *discriminant* d_F of a number field.

One wishes to recreate the notion of the prime numbers of $\mathbb{Z}$ within the ring of integers of an arbitrary number field. Unlike the rational integers, algebraic integers do not necessarily factor uniquely as a product of prime elements. For example in the ring of integers of $K = \mathbb{Q}(\sqrt{-5})$ the integer 6 factors as both $2 \cdot 3$ and $(1+\sqrt{-5})(1-\sqrt{-5})$. These four factors of 6 are all irreducible and hence prime elements of the ring $\mathcal{O}_K$. One of the most comforting features of the integers $\mathbb{Z}$ has been lost when passing to the more general integers of an arbitrary number field.

We are able to reclaim the concept of unique factorization by focusing on the ideals of a general ring of integers. Indeed an ideal $\mathfrak{a} \subset \mathcal{O}_K$ may be factored uniquely as a product of prime ideals [13, Theorem I.3.3]

$$\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{p}_i^{e_i}.$$

A prime of a number field K may be a (non-zero) prime ideal $\mathfrak{p}$ of the ring of integers O_K. Such primes are referred to as the *finite* or *non-archimedean* primes of K. The *infinite* primes of K are related to the aforementioned embeddings of K into $\mathbb{C}$.

2.2 Prime Ideals and Galois Extensions

If F and K are algebraic number fields with $F \subset K$ we refer to K as an *extension* of F. The *degree* of the extension K/F, written $[K : F]$, is the dimension of K considered as a vector space over F. One may consider the group of automorphisms of K that leave F fixed and denote it $\mathrm{Aut}(K/F)$. In general one has the inequality $\# \mathrm{Aut}(K/F) \leq [K : F]$. When this is an equality, one says that the extension is *Galois* and calls the group of automorphisms fixing F the Galois group of K/F. We will be primarily concerned with the situation where $\mathrm{Gal}(K/F)$ is an abelian group. In this case we often refer to the extension K/F as *abelian*.

If $\mathfrak{p}$ is a prime ideal of the ring $\mathcal{O}_F$, it is natural to consider the way that $\mathfrak{p}$ factors when considered as an ideal of $\mathcal{O}_K$. In the case of a Galois extension one has the factorization [13, page 55]

$$\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^e \mathfrak{P}_2^e \cdots \mathfrak{P}_g^e. \tag{2.1}$$

Here each $\mathfrak{P}_i$ is a prime ideal of $\mathcal{O}_K$ that lies above $\mathfrak{p}$. We often say that the ideal $\mathfrak{P}_i$ divides $\mathfrak{p}$ and write $\mathfrak{P}_i | \mathfrak{p}$. The natural number e is referred to as the *ramification index* of $\mathfrak{P}_i$ over $\mathfrak{p}$. The prime $\mathfrak{p}$ is said to ramify in the extension K/F if the power e is larger than one. Otherwise one says that $\mathfrak{p}$ is unramified.

The ring of integers of a number field is a Noetherian, integrally closed domain of Krull dimension one (i.e. a Dedekind Domain). It follows that its nonzero prime ideals are all maximal and can be shown to be of finite index. This implies that the quotients $\mathcal{O}_K/\mathfrak{P}_i$ and $\mathcal{O}_F/\mathfrak{p}$ are isomorphic to finite fields $\mathbb{F}_q$ and $\mathbb{F}_{q^f}$ respectively for some prime power q. The degree of the field extension $\Big[\big(\mathcal{O}_K/\mathfrak{P}_i\big) : \big(\mathcal{O}_F/\mathfrak{p}\big)\Big] = f(\mathfrak{P}_i/\mathfrak{p})$ is called the *inertia degree* of $\mathfrak{P}_i$ over $\mathfrak{p}$. Let $f(\mathfrak{P}_{\mathrm{i}}/\mathfrak{p}) = f_i$ In the case of a Galois extension one may prove that $f_1 = f_2 = \cdots = f_g$. One may also show that for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, the factorization (2.1) satisfies the fundamental equation $n = [K : F] = efg$. When $n = g$ one says that the prime $\mathfrak{p}$ *splits completely* in the extension K/F.

An *infinite prime* v of F is either a real embedding of F or a conjugate pair of complex embeddings of F. Suppose w is an infinite prime of K. Let i_v and i_w denote embeddings corresponding to v and w. One says that w lies above or divides v provided the restriction of i_w to F and i_v coincide. The infinite prime v is said to ramify in K/F if and only if it is a real prime and it has a complex prime lying above it in K.

For each prime $\mathfrak{P}$, finite or infinite, of the field K, one obtains a normalized absolute value $|\alpha|_{\mathfrak{P}}$ on K. If $\mathfrak{P}$ is a real infinite prime then this is the usual absolute value of the image of α under the corresponding embedding into $\mathbb{R}$. If $\mathfrak{P}$ is complex it is the square of the usual complex absolute value of the image. If $\mathfrak{P}$ is a finite prime then $\mathcal{N}(\mathfrak{P}) = |\mathcal{O}_K/\mathfrak{P}|$ is the *absolute norm* of $\mathfrak{P}$. One takes

$$|\alpha|_{\mathfrak{P}} = \mathcal{N}(\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}}(\alpha)}. \tag{2.2}$$

Here $\mathrm{ord}_{\mathfrak{P}}(\alpha)$ is the power of $\mathfrak{P}$ appearing in the factorization of the principal ideal $(\alpha)\mathcal{O}_K$.

2.3 Units, Class Groups and S-modification

Let U_F denote the units of the ring of integers $\mathcal{O}_F$. This is the set of all integers that possess a multiplicative inverse in $\mathcal{O}_F$. Let μ_F denote the group of roots of unity inside F. The structure of U_F as a multiplicative $\mathbb{Z}$-module is given by the following theorem of Dirichlet [13, Theorem I.7.4].

Theorem 2.3.1. *(Dirichlet's Unit Theorem) As a $\mathbb{Z}$-module, U_F is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times \mu_F$.*

Suppose $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t\}$ is a set of primes of F. We will always insist that S contains all infinite primes of F. Later we will also want S to contain all primes that ramify in the extension K/F and a certain number of splitting primes. Let $S_{\text{fin}} \subset S$ denote the set of all non-archimedean primes in S. We define the S-integers of F as follows.

$$\mathcal{O}_{F,S} = \{x \in F \mid \text{ord}_{\mathfrak{p}}(x) \geq 0 \quad \text{for all } \mathfrak{p} \notin S_{\text{fin}}\} \tag{2.3}$$

Let S_K be the set of all primes of K lying above those in S. The S-integers of K are given by the set

$$\mathcal{O}_{K,S} = \{x \in K \mid \text{ord}_{\mathfrak{P}}(x) \geq 0 \quad \text{for all } \mathfrak{P} \notin (S_K)_{\text{fin}}\}. \tag{2.4}$$

If the set S contains only the infinite primes, one reclaims the usual rings of integers of F and K. Adding finite primes to S has the effect of expanding the ring of integers by allowing elements having negative valuation at those finite places. One may consider the units of these new rings. We define the **S-units** of F, $U_{F,S}$, to be the set of all invertible elements of $\mathcal{O}_{F,S}$. Likewise the S-units of K are $U_{K,S} = \mathcal{O}_{K,S}^\times$. Dirichlet's Unit Theorem may be extended to inform us as to the $\mathbb{Z}$-module structure of S-units [13, Corollary I.11.7].

Theorem 2.3.2. *(Dirichlet) As a* $\mathbb{Z}$*-module one has the isomorphism*

$$U_{F,S} \cong \mathbb{Z}^{|S|-1} \times \mu_F.$$

A **fractional ideal** of a number field F is a (non-zero) finitely generated $\mathcal{O}_F$-submodule of F. For example any element $\alpha \in F$ generates a principal fractional ideal (α). The set of all fractional ideals $\mathcal{I}_F$ forms an abelian group under multiplication. We let P_F denote the subroup of $\mathcal{I}_K$ consisting of all principal fractional ideals. The quotient group

$$Cl_F = \mathcal{I}_K / P_K$$

is called the **ideal class group** of F. It is a finite abelian group that is the subject of a great deal of research. Its order is denoted by $|Cl_F| = h_F$ and is called the *class number* of F. When $h_F = 1$, one knows that every fractional ideal and hence every ideal of the ring $\mathcal{O}_F$ is principal. That is, the ring $\mathcal{O}_F$ is a principal ideal domain. As a principal ideal domain is a unique factorization domain, one sees that $h_F = 1$ implies that the elements of $\mathcal{O}_F$ factor uniquely into product of irreducible elements.

It is known that every ideal class $[\mathfrak{a}] \in Cl_F$ may be represented by a prime ideal of $\mathcal{O}_F$. We may define the S-modified class group as follows

Definition 2.3.3.

$$Cl_{F,S} = Cl_F / \langle [\mathfrak{p}] | \mathfrak{p} \in S_{\text{fin}} \rangle. \tag{2.5}$$

We set $h_{F,S} = |Cl_{F,S}|$ and refer to this as the S-class number of F.

2.4 Zeta-Functions

Stark's conjectures and their generalizations are concerned with certain complex valued functions called L-functions. The first example of an L-function is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{2.6}$$

One proves that the above sum defines an analytic function on the right half plane on which it is defined. One may prove that $\zeta(s)$ also has the Euler product expansion on this domain

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}. \tag{2.7}$$

The function $\zeta(s)$ admits a meromorphic continuation to the complex plane, has its only pole at $s = 1$, and satisfies the following functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \tag{2.8}$$

relating its value at s to its value at $1-s$. Here the term $\Gamma(s)$ is the well known gamma function.

Let K be an algebraic number field. The Riemann zeta function is the zeta function associated to the field $K = \mathbb{Q}$. The Dedekind zeta function of K, $\zeta_K(s)$ is first defined by the sum

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{\mathcal{N}(\mathfrak{a})^s}, \quad \Re(s) > 1. \tag{2.9}$$

Here the sum is taken over all integral ideals of $\mathcal{O}_K$. This provides a generalization of the Riemann zeta function and also satisfies the product expansion

$$\zeta_K(s) = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}}. \tag{2.10}$$

Here the product is taken over all *prime* ideals of $\mathcal{O}_K$. One shows that the Dedekind zeta function of a number field K admits a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation relating $\zeta_K(s)$ to $\zeta_K(1-s)$. It has a single simple pole at $s = 1$

with residue

$$\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K|d_K|^{1/2}}. \tag{2.11}$$

Here w_K is the number of roots of unity in K and R_K is the regulator of K (the absolute value of the determinant of a matrix involving logarithms of absolute values of fundamental units of K). The invariant d_K is the discriminant of K. The residue of $\zeta_K(s)$ at $s = 1$ encodes a great deal of information about the associated number field. This formula for the residue is called the **analytic class number formula**.

The functional equation enables us to translate the analytic class number formula to focus instead on the behavior of $\zeta_K(s)$ about $s = 0$. One finds that $\zeta_K(s)$ has a zero of order $r = r_1 + r_2 - 1$ at $s = 0$ and

$$\lim_{s\to 0} s^{-r}\zeta_K(s) = -\frac{h_K R_K}{w_K}. \tag{2.12}$$

This formula encodes the same information and is simpler. Stark preferred working with L-functions at $s = 0$ and we follow him here.

If S is a set of primes of K containing all of the infinite primes, one may define the S-incomplete Dedekind zeta function by simply removing the Euler factors corresponding to the primes in S_{fin}. One obtains the new zeta function $\zeta_{K,S}(s)$ which satisfies its own version of the analytic class number formula

$$\lim_{s\to 0} s^{1-|S|}\zeta_{K,S}(s) = -\frac{h_{K,S} R_{K,S}}{w_K}. \tag{2.13}$$

which provides information regarding the S-class number of K.

2.5 L-Functions

All of the zeta functions we have seen involved sums of infinitely many ratios whose numerators were all one. The first examples of L-functions that deviate from this pattern are Dirichlet L -Functions. Let $m \geq 1$ be an integer. Let χ be a Dirichlet character modulo m. That is χ is a group homomorphism $\chi : (\mathbb{Z}/m\mathbb{Z})^* \longrightarrow S^1 \subset \mathbb{C}$. If $m|n$ then χ also defines a Dirichlet character modulo n via the composition

$$(\mathbb{Z}/n\mathbb{Z})^* \longrightarrow (\mathbb{Z}/m\mathbb{Z})^* \xrightarrow{\chi} S^1$$

where the first arrow is the natural map taking $a \pmod{n}$ to $a \pmod{m}$. We will choose m to be minimal. That is we will suppose that all of our characters are *primitive* and call $m = f_\chi = f$ the *conductor* of χ. It will be convenient for us to extend our characters to all of $\mathbb{Z}$ by letting $\chi(k) = \chi(k \pmod{f})$ if $(f, k) = 1$ and zero otherwise.

We define the Dirichlet L-function for the character χ with conductor f by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}, \quad \Re(s) > 1. \tag{2.14}$$

It can be extended to an analytic function (when $\chi \neq 1$) on $\mathbb{C}$ which satisfies a functional equation relating $L(s,\chi)$ to $L(1-s,\bar{\chi})$. Let $\delta = 0$ if $\chi(-1) = 1$ and $\delta = 1$ otherwise. The functional equation may be written

$$\Gamma(s)\cos\left(\frac{\pi(s-\delta)}{2}\right)L(s,\chi) = \frac{\tau(\chi)}{2i^\delta}\left(\frac{2\pi}{f}\right)^s L(1-s,\bar{\chi}). \tag{2.15}$$

Here $\tau(\chi) = \sum_{a=1}^{f} \chi(a) e^{2\pi a/f}$ is a Gauss sum.

It will be profitable for us in the future to recast the Dirichlet character χ in a more Galois-theoretic setting. Let $\zeta_f \in \mathbb{C}$ be a primitive f-th root of unity. Let p be a prime number that is relatively prime to f. The group $(\mathbb{Z}/f\mathbb{Z})^*$ is isomorphic to the Galois group G of the extension $\mathbb{Q}(\zeta_f)/\mathbb{Q}$ via the correspondence

$$p \pmod{f} \longmapsto \phi_p.$$

The automorphism $\phi_p \in G$ is defined by $\phi_p(\zeta) = \zeta^p$ for each root of unity $\zeta \in \mathbb{Q}(\zeta_f)$. The map is defined on all of $(\mathbb{Z}/f\mathbb{Z})^*$ by extending multiplicatively. As such we may consider the Dirichlet character χ a character of the Galois group G. This leads us to the setting of Artin's L-functions. Dirichlet L-functions are special cases of these.

Suppose K/F is a Galois extension of number fields with Galois group $G = \mathrm{Gal}(K/F)$. Let V be a finite dimensional vector space over $\mathbb{C}$. Let ρ be a representation of G on V. That is, ρ is a homomorphism

$$\rho : G \longrightarrow GL(V).$$

Denote by χ the character associated to ρ. That is, $\chi(s) = \mathrm{Tr}(\rho(s))$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ and $\mathfrak{P}$ a prime of $\mathcal{O}_K$ lying above $\mathfrak{p}$. We associate two subgroups of the Galois group to the prime $\mathfrak{P}$.

Definition 2.5.1. The **decomposition group** associated to the prime $\mathfrak{P}$ is

$$G_{\mathfrak{P}} = \{s \in G \mid s \cdot x \in \mathfrak{P} \ \forall x \in \mathfrak{P}\}.$$

The **inertia group** associated to $\mathfrak{P}$ is

$$I_{\mathfrak{P}} = \{s \in G \mid s \cdot x \equiv x \pmod{\mathfrak{P}} \ \forall x \in \mathcal{O}_K\}.$$

Two elements $x, \alpha \in \mathcal{O}_K$ satisfy the congruence $x \equiv \alpha \pmod{\mathfrak{P}}$ when $x - \alpha \in \mathfrak{P}$. There is a canonical isomorphism

$$G_{\mathfrak{P}}/I_{\mathfrak{P}} \xrightarrow{\sim} \mathrm{Gal}\Big(\big(\mathcal{O}_K/\mathfrak{P}\big)/\big(\mathcal{O}_F/\mathfrak{p}\big)\Big)$$

from the quotient onto the Galois group of the residue class field extensions. As the latter group is the Galois group of a finite extension of finite fields, the quotient is necessarily cyclic. Let $\sigma_{\mathfrak{P}}$ denote the generator of the quotient $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ whose image in $\mathrm{Gal}\Big((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_F/\mathfrak{p})\Big)$ is the q-th power map where $q = \mathcal{N}(\mathfrak{p})$. Each element of the coset $\sigma_{\mathfrak{P}} I_{\mathfrak{P}}$ is called a **Frobenius automorphism** associated to $\mathfrak{P}$.

Suppose that $\mathfrak{p}$ is unramified in K/F. Then $I_{\mathfrak{P}}$ is trivial and we may associate to $\mathfrak{P}|\mathfrak{p}$ a unique Frobenius automorphism. The conjugacy class of $\sigma_{\mathfrak{P}}$ in G does not depend on the choice of the prime $\mathfrak{P}|\mathfrak{p}$ [11, III.2.1]. We extend this to an arbitrary fractional ideal $\mathfrak{a} \in \mathcal{I}_K$. If $\mathfrak{a} = \frac{\mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_l}{\mathfrak{P}_{l+1}\mathfrak{P}_{l+2}\cdots\mathfrak{P}_{l+m}}$ where each $\mathfrak{P}_i$ is a prime ideal of $\mathcal{O}_K$ we assign to $\mathfrak{a}$ a "Frobenius" element $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{P}_1}\sigma_{\mathfrak{P}_2}\cdots\sigma_{\mathfrak{P}_l}\sigma_{\mathfrak{P}_{l+1}}^{-1}\sigma_{\mathfrak{P}_{l+2}}^{-1}\cdots\sigma_{\mathfrak{P}_{l+m}}^{-1}$.

Let S be a finite set of primes of F containing all infinite primes and all finite primes that ramify in K/F. Let $\mathcal{I}_K^S$ be the group of all fractional ideals of K prime to the finite primes in S. The map

$$\mathrm{Frob} : \mathcal{I}_K^S \longrightarrow G$$
$$\mathfrak{a} \longmapsto \sigma_{\mathfrak{a}}$$

assigns to each fractional ideal of K that is prime to S an element of the Galois group. It is known as the **Artin map**.

Assume that our extension K/F is abelian and the prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ is unramified in K/F. The Frobenius element associated to the ideal $\mathfrak{P}|\mathfrak{p}$ depends simply on the ideal $\mathfrak{p}$ lying below it.

Definition 2.5.2. For each $\chi \in \hat{G}$ we may define the S-imprimitive **Artin L-function**

as

$$L_{K/F,S}(s,\chi) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F \\ (\mathfrak{a},S)=1}} \frac{\chi(\sigma_{\mathfrak{a}})}{(\mathcal{N}\mathfrak{a})^s} = \prod_{\substack{\mathfrak{p}\subset\mathcal{O}_F \\ \mathfrak{p}\notin S}} \left(1-\chi(\sigma_{\mathfrak{p}})(\mathcal{N}\mathfrak{p})^{-s}\right)^{-1}. \tag{2.16}$$

The Artin L-function associated to a non-abelian extension K/F is more difficult to define. In the sequel all of our Galois extensions will be abelian. The interested reader should see Neukirch [13, VII.10.1] for the general definition. The Artin L-series defines an analytic function on the half plane $\Re(s) > 1$. One proves that it has an analytic continuation to $\mathbb{C}\setminus\{1\}$ and satisfies a functional equation relating its value at s to its value at $s-1$. These L-functions also exhibit the following functorial properties [13, Proposition VII.10.4]. Let M denote a field intermediate to K and F, and L a field containing the other three that is Galois over F. That is we have the chain of containments $F \subset M \subset K \subset L$.

Proposition 2.5.3. *(i) For the principal character $\chi = \mathbf{1}$, one has*

$$L_{K/F,S}(s,\mathbf{1}) = \zeta_{F,S}(s).$$

(ii) If χ, $\chi' \in \widehat{G}$, then

$$L_{K/F,S}(s,\chi+\chi') = L_{K/F,S}(s,\chi)L_{K/F,S}(s,\chi').$$

(iii) If $\chi \in \widehat{G}$ one may view χ as a character on the larger Galois group Gal(L/F) by composition with the quotient map. One then has

$$L_{L/F,S}(s,\chi) = L_{K/F,S}(s,\chi).$$

(iv) If χ is a character of the group Gal(K/M) and S_M denotes the primes of M that

divide the primes in S then one has

$$L_{K/M,S_M}(s,\chi) = L_{K/F,S}(s,\chi*).$$

Here $\chi*$ denotes the character on G *induced* from χ. When one forms the induced character $\mathbf{1}^*$ from the trivial character $\mathbf{1}$ of the trivial subgroup $\{\mathbf{1}\} < \mathrm{Gal}(K/F) = G$ one obtains the character associated to the left regular representation of G. This character r_G decomposes as the sum

$$r_G = \sum_{\chi} \chi$$

taken over all irreducible characters of G. Combining this with parts (i), (ii), and (iv) of the proposition give us the following result.

Corollary 2.5.4. *One has*

$$\zeta_{K,S}(s) = \zeta_{F,S}(s) \prod_{\chi \neq \mathbf{1}} L_{K/F,S}(s,\chi). \tag{2.17}$$

This corollary states that one can factor the Dedekind zeta function of a number field as a product of Artin L-functions.

2.6 Stark's Refined Abelian Conjecture

In a series of four papers written in the 1970's and 1980's Harold Stark set forth his conjectures. In their most basic form Stark's conjectures propose an "analytic class number formula" for Artin L-functions. A vague form of his general conjecture is stated in the second paper in the series [21].

Conjecture 2.6.1 (($A(K/F)$)**.** *Suppose K/F is a Galois extension of number fields, χ is a character of Gal(K/F) not containing a copy of the trivial character, and a is the* *order of vanishing of $L_{K/F}(s,\chi)$ at $s=0$. Then one has*

$$\lim_{s\to 0} s^{-a} L_{K/F}(s,\chi) = \theta(\chi)R(\chi)$$

where $\theta(\chi)$ is an algebraic number and $R(\chi)$ is the determinant of an a by a matrix whose entries are linear forms (with algebraic coefficients) in logarithms of absolute values of units belonging to K and its conjugate fields.

Combining this conjecture with the factorization of the Dedekind zeta function (Corollary 2.5.4) yields the conjectural factorization of the analytic class number formula

$$-\frac{h_K R_K}{w_K} = \prod_{\chi} \theta(\chi)R(\chi). \tag{2.18}$$

Stark proved this conjecture for all rational characters. In the fourth paper in the series [22], Stark unveiled a refined abelian conjecture for order of vanishing $a=1$. In this case the algebraic factor $\theta(\chi)$ is conjectured to be a rational number whose denominator would be bounded. Let S be a set of primes of the base field F that contains all ramified and infinite primes. For simplicity we will assume that $|S| \geq 3$.

Conjecture 2.6.2 (St($K/F,S$))**.** *Suppose that S contains a prime v that splits completely in the extension K/F. Fix a prime w of K above v. Then there exists a unique element $\epsilon \in U_{K,S}$ satisfying:*

(1) for each $\chi \in \widehat{G}$,

$$L'_{K/F,S}(0,\chi) = -\frac{1}{w_K}\sum_{\sigma\in G} \chi(\sigma)\log|\epsilon^{\sigma}|_w$$

and

(2) the extension

$$K\left(\epsilon^{1/w_K}\right)/F$$

is abelian.

The conjecture has an alternate formulation in the case where S contains only two primes. In this case it is known to be true unconditionally [23, Proposition 3.10]. When the conjecture is known, it predicts that a single S-unit, known as a *Stark unit*, simultaneously evaluates the first derivatives at $s = 0$ of all Artin L-functions associated to K/F. Stark was able to prove his conjecture in the case where $F = \mathbb{Q}$ or F is an imaginary quadratic field.

Higher order refined abelian conjectures will unfortunately not predict the existence of a single Stark unit. Instead the r-th order conjectures of Rubin [18] and Popescu [16] predict that the r-th derivatives at $s = 0$ of the L functions may be evaluated upon applying a special *regulator map* to an element of the r-th exterior power of the S-unit group (tensored with $\mathbb{Q}$).

2.7 Exterior Products

Let R be a commutative ring with unity and M and N be (left) R-modules. A map $\phi : M^n \longrightarrow N$ is said to be an n-multilinear map if it is linear in each of its components. It is called alternating if $\phi(m_1, m_2, \ldots, m_n) = 0$ whenever $m_i = m_j$ for some $i \neq j$. When $M = R^n$, an alternating n-multilinear map $D : M^n \longrightarrow R$ is a constant multiple of the determinant of the matrix with columns given by the n-tuple $(r_1, r_2, \ldots, r_n)$ [5, page 437]. Due to the nature of the determinant map, one wishes to define a module through which multilinear alternating maps factor. The n^{th} exterior product $\bigwedge_R^n M$ can be defined as the R-module through which each n-multilinear map factors uniquely. When the ring R is clear we will often omit the subscript to avoid clutter. The module $\bigwedge^n M$ is isomorphic to the quotient of $\underbrace{M \otimes M \otimes \cdots \otimes M}_{n-\text{times}}$ by the submodule generated by all elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_n$ satisfying $m_i = m_j$ for some $i \neq j$.

If $M \cong R^n$ then $\bigwedge^n M$ is a free R module of rank 1 [8, A.2.3(d)]. Any R-module homomorphism $\phi : M \longrightarrow N$ induces a homomorphism $\bigwedge^n(\phi) : \bigwedge^n M \longrightarrow \bigwedge^n N$ in the obvious fashion. A special case of this is when $M = N$ is a free R-module of rank n. In this case if $m_1, m_2, \ldots, m_n$ gives a basis for M over R then $m_1 \wedge m_2 \wedge \cdots \wedge m_n$ spans the 1-dimensional free R-module $\bigwedge^n M$. Given any R-module homomorphism $\phi : M \longrightarrow M$ one obtains an endomorphism $\bigwedge^n(\phi) : \bigwedge^n M \longrightarrow \bigwedge^n M$ which necessarily takes the basis element $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ to a constant $D(\phi) \in R$ times itself. This constant is the determinant of the linear mapping ϕ. One sees that upon passage to the n-th exterior product of a free R-module of rank n, a homomorphism $\phi \in \mathrm{Hom}_R(M, M)$ is taken to the (1-dimensional) linear map $D(\phi) : \bigwedge^n M \longrightarrow \bigwedge^n M$. This explains why $\bigwedge^n M$ is often referred to as the "determinant space" of M in the literature.

Let us now suppose that G is an abelian group and that M has the structure of a $\mathbb{Z}[G]$-module. What follows is found in the paper of Rubin [18]. Let $\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ denote the set of all $\mathbb{Z}[G]$-module homomorphisms from M into $\mathbb{Z}[G]$. Let $\phi \in \mathrm{Hom}(M, \mathbb{Z}[G])$ and let r be a natural number. The map ϕ induces a $\mathbb{Z}[G]$-module homomorphism from $\bigwedge^r_{\mathbb{Z}[G]} M$ to $\bigwedge^{r-1}_{\mathbb{Z}[G]} M$ by taking

$$m_1 \wedge \cdots \wedge m_r \longmapsto \sum_{i=1}^{r} (-1)^{i+1} \phi(m_i)\, m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \cdots \wedge m_r$$

and extending by linearity. We abusively refer to the resulting homomorphism as ϕ. If we repeat this process k times for $1 \leq k \leq r$ and $\phi_1, \ldots \phi_k \in \mathrm{Hom}(M, \mathbb{Z}[G])$ we obtain a map

$$\bigwedge_{\mathbb{Z}[G]}^{k} \mathrm{Hom}(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}\left(\bigwedge_{\mathbb{Z}[G]}^{r} M, \bigwedge_{\mathbb{Z}[G]}^{r-k} M\right)$$

$$\phi_1 \wedge \cdots \wedge \phi_r \longmapsto \phi_k \circ \cdots \circ \phi_1.$$

When $k = r$ one obtains a determinant

$$(\phi_1 \wedge \cdots \wedge \phi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\phi_i(m_j)).$$

In his dissertation [10, Lemma 5.3] Hayward provided a rigorous proof of the following result.

Lemma 2.7.1. *If* $\Phi = \phi_1 \wedge \cdots \wedge \phi_k$ *and* $\boldsymbol{m} = m_1 \wedge \cdots \wedge m_r$ *then*

$$\Phi(\boldsymbol{m}) = \sum_{\substack{\sigma \in S_r \\ \sigma(k+1) < \cdots < \sigma(r)}} sign(\sigma) \left(\prod_{j=1}^{k} \phi_j\left(m_{\sigma(j)}\right) \right) m_{\sigma(k+1)} \wedge \cdots \wedge m_{\sigma(r)}.$$

We will be most concerned with the case when $k = r - 1$. In this case the lemma gives

$$\phi_1 \wedge \cdots \wedge \phi_{r-1}(m_1 \wedge \cdots \wedge m_r) = \sum_{1 \leq k \leq r} (-1)^{k+1} \det_{j \neq k}(\phi_i(m_j)) \cdot m_k.$$

2.8 Extension of Stark's Refined Abelian Conjecture

Stark's refined abelian conjecture $\mathrm{St}(K/F, S)$ is trivially satisfied when all L-functions associated to the extension K/F and the set S vanish to order r larger than 1 at $s = 0$. One may take $\epsilon = 1$ and see that the two conditions are satisfied. This compelled Rubin to formulate a refined "Stark type" conjecture over $\mathbb{Z}$ in his 1994 paper [18]. This conjecture, $B(K/F, S, T, r)$, is an integral refinement of Conjecture 2.6.1 for extensions where $L_{K/F,S}(s, \chi)$ vanishes to order r at $s = 0$ for all $\chi \in \widehat{G}$. It predicts that an element ϵ inside $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathbb{Z}[G]} U_{S,K}$ simultaneously evaluates all r-th derivatives of L-functions upon applying a special regulator function to it. The conjecture also predicts that the element ϵ satisfies certain integrality properties. That is to say there is an explicit bound for the denominators associated with ϵ.

Rubin's conjecture uses an auxiliary "smoothing" set of primes T, introduced by Gross [9], of the base field, satisfying $T \cap S = \emptyset$, that modifies the class group, unit group, and L-functions associated to the extension. The associated (S,T)-modified L-functions are defined by the equation

$$L_{K/F,S,T}(s,\chi) = \left(\prod_{v \in T}(1 - \chi(\sigma_v) \cdot (Nv)^{1-s})\right) L_{K/F,S}(s,\chi). \tag{2.19}$$

The factors clearly don't affect the functions' behavior at $s = 0$ with the exception of multiplying by a constant. The (S,T)-modified units of K are given by the set

$$U_{K,S,T} = \{u \in U_{K,S} \mid u \equiv 1 \pmod{w}, \forall\, w \in T_K\}. \tag{2.20}$$

The (S,T)-modified class group is given by

$$A_{K,S,T} = \frac{\{\text{fractional ideals of } \mathcal{O}_{K,S} \text{ coprime to } T_K\}}{\{\alpha\mathcal{O}_{K,S} \mid \alpha \equiv 1 \pmod{w}, \forall\, w \in T_K\}}. \tag{2.21}$$

The set T is chosen so that $\mu_{K,S,T}$, the torsion group of $U_{K,S,T}$, is trivial. This choice removes the denominator from the right hand side of the analytic class formula

$$\zeta_{K,S,T}(s) \equiv -h_{K,S,T}R_{K,S,T}s^r \pmod{s^{r+1}}. \tag{2.22}$$

When Rubin's conjecture is restricted to the situation where $r = 1$ one obtains Stark's refined abelian conjecture. The abelian condition is realized by allowing the auxiliary set of primes T to vary over all possible choices [17, Lemma 2.2.3].

Rubin proved conjecture $B(K/F,S,T,r)$ in the special cases where the set S contains more than r primes that split completely in the extension K/F and where $K = F$. The case $r = 0$ is known trivially if F is complex and by a theorem of Deligne and Ribet in the case that F is real. Rubin also proves the conjecture in full when K/F is a quadratic extension. Rubin's conjecture is known to be true in the case of a function field (i.e. $\mathrm{char}(F) = p > 0$) for the case $r = 1$ by work of Deligne and Hayes [17]. Popescu has made great strides in proving conjecture $B(K/F, S, T, r)$ in the function field setting for arbitrary order of vanishing r [17]. In characteristic 0, Conjecture $B(K/F, S, T, 1)$ is known when either $F = \mathbb{Q}$ or a quadratic imaginary field by the work of Stark. Sands, Dummit, and Tangedal have proved $B(K/F, S, T, 1)$ for almost all multiquadratic extension K/F [6].

Popescu introduced a weaker refined abelian conjecture in 2002 [16] that could be stated using one set of primes S and satisfied a base-change property that can not currently be shown for conjecture $B(K/F, S, T, r)$. Popescu's conjecture $C(K/F, S, r)$ has been shown to follow from conjecture $B(K/F, S, T, r)$ [16]. Sands proved conjecture $C(K/k, S, r)$ for multiquadratic extensions with restrictions on the set S and the extension K/F [20]. In the sequel we aim to extend these results and identify new settings where Popescu's conjecture is known to hold.

Recently Burns and his collaborators have formulated a vast generalization of the refined abelian conjectures. The Equivariant Tamagawa Number Conjecture (ETNC) applies to non-abelian L-functions. Burns has proven [2, Corollary 4.1] that when known to be true, ETNC implies the truth of Rubin's conjecture and therefore Popescu's conjecture as well. This allows one to establish conjectures $B(K/F, S, T, r)$ and $C(K/F, S, r)$ for a wide class of extensions in which they were previously unknown. These results require a somewhat restrictive choice of set S and we work to establish the conjectures without using ETNC.

Chapter 3

A Conjecture of Popescu

3.1 Statement of the Conjectures

Let K/F be an abelian extension of number fields with Galois group G and let r be a nonnegative integer. Let S be a finite set of places of F. We will assume from now on that the data $(K/F, S, r)$ satisfy the following set of hypotheses.

Hypotheses 3.1.1 (H_r). *1. S contains all places that ramify in K/F and all infinite places of F.*

2. $\#S \geq r+1$

3. S contains at least r places that split completely in K/F.

Popescu's conjecture is concerned with the first nonvanishing coefficient in the power series expansion at $s = 0$ of the S-imprimitive Artin L-functions (Definition 2.5.2) associated to characters $\chi \in \widehat{G}$. By Tate [23, I.3.4], the function $L_{K/F,S}(s, \chi)$ vanishes to degree $r_{\chi,S}$, where

$$r_{\chi,S} = \begin{cases} \#S - 1, & \text{if } \chi = \mathbf{1}; \\ \#\{v \in S : \chi(G_w) = \{1\} \text{ for all } w \mid v\}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Here G_w is the decomposition group (Definition 2.5.1) for the place w of K. Formula 3.1 shows that if the extension K/F and set S satisfy Hypotheses H_r, then $L_{K/F,S}(s,\chi)$ vanishes to order at least r.

Definition 3.1.2. The **Stickelberger function** is the map $\Theta_{K/F,S} : \mathbb{C} \longrightarrow \mathbb{C}[G]$ defined by

$$\Theta_{K/F,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/F,S}(s,\chi) \cdot e_{\bar{\chi}}$$

where $e_\chi = \frac{1}{\#G} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathbb{C}[G]$ is the idempotent corresponding to χ.

Definition 3.1.3. The **Stickelberger element** for the data $(K/F, S, r)$ is

$$\Theta^{(r)}_{K/F,S}(0) = \lim_{s \to 0} \frac{\Theta_{K/F,S}(s)}{s^r} \in \mathbb{C}[G].$$

The set S contains r places that split completely in K/F so that $\lim_{s\to 0} s^{-r} L_{K/F,S}(s,\chi) \neq \infty$ for each $\chi \in \widehat{G}$. Therefore $\Theta^{(r)}_{K/F,S}(0)$ is an element in the group ring $\mathbb{C}[G]$.

Let $V = (v_1, v_2, \ldots, v_r)$ be an r-tuple of places of S that split completely in K/F. Let $W = (w_1, w_2, \ldots, w_r)$ be an r-tuple of places of S_K such that $w_i \mid v_i$ for each i.

Definition 3.1.4. The W**-regulator map** $R_W : \bigwedge^r_{\mathbb{Z}[G]} U_{K,S} \longrightarrow \mathbb{C}[G]$ is given by

$$R_W : (u_1 \wedge u_2 \wedge \cdots \wedge u_r) \longmapsto \det_{1 \le i,j \le r} \big(-\sum_{\sigma \in G} \log |u_j|_{w_i^\sigma} \cdot \sigma \big)$$

for any $u_1, u_2, \ldots, u_r \in U_{K,S}$.

Upon extending by $\mathbb{C}$-linearity to all of $\mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^r U_{K,S}$, one obtains a $\mathbb{C}[G]$-module homomorphism that we will also call R_W.

From now on, when k is a field and M is a $\mathbb{Z}[G]$-module we will write the tensor $k \otimes_{\mathbb{Z}} M$ as kM. Let $R \subseteq \mathbb{C}$ be a ring and suppose M is a $R[G]$-module. Define the

$R[G]$-submodule

$$M_{r,S} = \{m \in M \,|\, e_\chi m = 0 \in \mathbb{C} \otimes_R M \text{ for all } \chi \in \hat{G} \text{ such that } r_{\chi,S} > r\}. \tag{3.2}$$

For any $m \in M$, let $\widetilde{m}$ denote the image of m in $\mathbb{Q}M$ under the natural map $M \to \mathbb{Q}M$ and let $\widetilde{M}$ denote the image of M under this map.

Definition 3.1.5. Let

$$U^{\mathrm{ab}}_{K/F,S} = \{u \in U_{K,S} | K(u^{1/w_K})/F \text{ is abelian}\}.$$

Theorem 3.1.6. *The restriction of R_W to the submodule $(\mathbb{C} \bigwedge^r U_{K,S})_{r,S}$ gives a $\mathbb{C}[G]$-module isomorphism between $(\mathbb{C} \bigwedge^r U_{K,S})_{r,S}$ and $\mathbb{C}[G]_{r,S}$.*

Proof. We refer the reader to Remark 1 of Popescu's paper [16]. □

We have defined the Stickelberger element $\Theta^{(r)}_{K/F,S}(0) \in \mathbb{C}[G]_{r,S}$. If for some character $\psi \in \widehat{G}$ we apply ψ to $\Theta^{(r)}_{K/F,S}(0)$ we obtain

$$\psi\left(\Theta^{(r)}_{K/F,S}(0)\right) = \psi\left(\lim_{s\to 0} s^{-r} \sum_{\chi \in \widehat{G}} L_{K/F,S}(s,\chi)\cdot e_{\bar{\chi}}\right) = \lim_{s\to 0} s^{-r} \sum_{\chi \in \widehat{G}} L_{K/F,S}(s,\chi)\cdot \psi\left(e_{\bar{\chi}}\right)$$

$$= \lim_{s\to 0} s^{-r} L_{K/F,S}(s,\bar{\psi}) \in \mathbb{C}.$$

The complex number $\psi\left(\Theta^{(r)}_{K/F,S}(0)\right)$ will be zero whenever $r_{\bar{\psi},S} > r$. By Theorem 3.1.6, the mapping R_W takes exactly one element of $(\mathbb{C} \bigwedge^r U_{K,S})_{r,S}$ to $\Theta^{(r)}_{K/F,S}(0)$. Let $\epsilon_{K/F,S,W} \in (\mathbb{C} \bigwedge^r U_{K,S})_{r,S}$ be the preimage under R_W of $\Theta^{(r)}_{K/F,S}(0)$. To summarize, we have for all $\chi \in \widehat{G}$ that

$$\chi\left(R_W\left(\epsilon_{K/F,S,W}\right)\right) = \chi\left(\Theta^{(r)}_{K/F,S}(0)\right) = \lim_{s\to 0} s^{-r} L_{K/F,S}(0,\bar{\chi}).$$

Applying R_W to the element $\epsilon_{K/F,S,W}$ evaluates the r-th coefficient of the power series at $s = 0$ for all of the S-incomplete Artin L-Functions simultaneously. Popescu's Conjecture predicts that this element $\epsilon_{K/F,S,W}$ lies in a certain submodule of $(\mathbb{C}\bigwedge^r U_{K,S})_{r,S}$.

Let $U_{K,S}^*$=$\mathrm{Hom}_{\mathbb{Z}[G]}(U_{K,S}, \mathbb{Z}[G])$ denote the dual $\mathbb{Z}[G]$-module of $U_{K,S}$. For each $(r-1)$-tuple $(\phi_1, \phi_2, \ldots, \phi_{r-1}) \in \left(U_{K,S}^*\right)^{r-1}$ one obtains a $\mathbb{Z}[G]$-module homomorphism

$$\phi_1 \wedge \cdots \wedge \phi_{r-1} : \mathbb{C}\bigwedge^r U_{K,S} \longrightarrow \mathbb{C}\bigwedge^1 U_{K,S} = \mathbb{C}U_{K,S}.$$

given explicitly given by Lemma 2.7.1 as

$$(\phi_1 \wedge \cdots \wedge \phi_{r-1})(u_1 \wedge \cdots \wedge u_r) = \sum_{1 \le k \le r} (-1)^{k+1} \det_{j \neq k} (\phi_i(u_j)) \cdot u_k. \tag{3.3}$$

We use these maps to define two subsets of $\mathbb{C}\bigwedge^r U_{K,S}$.

Definition 3.1.7 (Popescu's Lattice)**.**

$$\begin{aligned}\bigwedge\nolimits_0^r U_{K,S} =&\{\epsilon \in (\mathbb{Q}\bigwedge^r U_{K,S})_{r,S} : \\ &(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\epsilon) \in \frac{1}{w_K}\widetilde{U_{K/F,S}^{\mathrm{ab}}}, \forall \phi_1, \ldots, \phi_{r-1} \in U_{K,S}^*\}.\end{aligned}$$

Definition 3.1.8 (Sands' Lattice)**.**

$$\begin{aligned}\bigwedge\nolimits_1^r U_{K,S} =&\{\epsilon \in (\mathbb{Q}\bigwedge^r U_{K,S})_{r,S} | \\ &(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\epsilon) \in \frac{1}{w_K}\widetilde{U_{K,S}}, \forall \phi_1, \ldots, \phi_{r-1} \in U_{K,S}^*\}.\end{aligned}$$

Let us now state the conjectures that are the focus of our study.

Conjecture 3.1.9 (Popescu $C(K/F,S,r)$). *Suppose the extension K/F, the set S, and the positive integer r satisfy hypotheses H_r. Then $\epsilon_{K/F,S,W} \in \bigwedge_0^r U_{K,S}$.*

Conjecture 3.1.10 ($C'(K/F,S,r)$). *Under the same assumptions as above one has $\epsilon_{K/F,S,W} \in \bigwedge_1^r U_{K,S}$.*

Conjecture $C'(K/F,S,r)$ may be thought of as Popescu's conjecture without the abelian condition. Conjecture $C(K/F,S,r)$ implies conjecture $C'(K/F,S,r)$. The evaluator $\epsilon_{K/F,S,W}$ in the conjectures depends on the choice of the r-tuple W. It turns out that this dependence is not problematic and if conjecture $C(K/F,S,r)$ holds for one choice of W then it holds for all others [17, Remark 2]. In the case where $r=1$, we have simply $\bigwedge_0^r U_{L,S} = \frac{1}{w_K}\left(\widetilde{U^{\text{ab}}_{K/F,S}}\right)_{1,S}$. It follows that conjecture $C(K/F,S,1)$ is equivalent to Stark's own refined abelian conjecture. Popescu's conjecture is a true extension of Stark's conjecture.

In order to investigate these conjectures we will need more information as to the structure of the sets $\bigwedge_0^r U_{K,S}$ and $\bigwedge_1^r U_{K,S}$.

Theorem 3.1.11. *$\bigwedge_0^r U_{K,S}$ and $\bigwedge_1^r U_{K,S}$ are $\mathbb{Z}[G]$-modules.*

Proof. Let $\epsilon_1, \epsilon_2 \in \bigwedge_1^r U_{K,S}$. Let $\phi_1, \phi_2, \ldots, \phi_{r-1} \in U^*_{K,S}$. Let $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_{r-1}$ as defined in formula 3.3. We need to show that $\phi(\epsilon_1 + \epsilon_2) \in \frac{1}{w_K}\widetilde{U_{K,S}}$. The map ϕ is a $\mathbb{Z}[G]$-module homomorphism so in $\mathbb{Q}U_{K,S}$ one has

$$\phi(\epsilon_1 + \epsilon_2) = \phi(\epsilon_1)\phi(\epsilon_2) = \widetilde{u}\widetilde{v}. \tag{3.4}$$

Since $\widetilde{u}, \widetilde{v} \in \frac{1}{w_K}\widetilde{U_{K,S}}$ they are the images in $\mathbb{Q}U_{K,S}$ of w_K-th roots of S-units, say u and v. As the product of u and v is again a w_K-th root of an S-unit, we have $\phi(\epsilon_1+\epsilon_2) \in \frac{1}{w_K}\widetilde{U_{K,S}}$.

Let $\sigma \in G$ and $\epsilon \in \bigwedge_1^r U_{K,S}$. We need to show that $\sigma \cdot \epsilon \in \bigwedge_1^r U_{K,S}$. Since ϕ is a

$\mathbb{Z}[G]$-module homomorphism one has

$$\phi(\sigma \cdot \epsilon) = \sigma \cdot \phi(\epsilon) \tag{3.5}$$

The right-hand side of equation 3.5 is an element of $\frac{1}{w_K}\widetilde{U_{K/F,S}}$ since $U_{K,S}$ is itself a $\mathbb{Z}[G]$-module. It follows that $\bigwedge_1^r U_{K,S}$ is a $\mathbb{Z}[G]$-module.

Now let $\epsilon_1, \epsilon_2 \in \bigwedge_0^r U_{K,S}$. We already have seen that $\phi(\epsilon_1)$ and $\phi(\epsilon_2)$ are the image in $\mathbb{Q}U_{K,S}$ of w_K-th roots u and v of S-units. We know that the extensions $K(u)$ and $K(v)$ are abelian extensions of F. One has $K(uv) \subset K(u)K(v)$. The compositum of abelian extensions is abelian. It follows that $K(uv)/F$ is an abelian extension. This shows that $\phi(\epsilon_1 + \epsilon_2) \in \frac{1}{w_K}\widetilde{U^{\text{ab}}_{K/F,S}}$.

Let $\epsilon \in \bigwedge_0^r U_{K,S}$ and $\sigma \in G$. Now we must show that $\sigma \cdot \epsilon \in \bigwedge_0^r U_{K,S}$. We have that $\phi(\epsilon) = \widetilde{u}$ is the image of the w_K-th root u in $\mathbb{Q}U_{K,S}$. We know that the extension $K(u)/F$ is Galois. Let $\widetilde{\sigma} \in \text{Gal}(K(u)/F)$ be a lift of σ. The extension $K(u^{\widetilde{\sigma}})$ is identical to $K(u)$ and is therefore Galois over F as well. It follows that $\sigma \cdot \epsilon \in \bigwedge_0^r U_{K,S}$. □

3.2 Evaluators for Relative Quadratic and Multiquadratic Extensions

Let L/F be a multiquadratic extension of number fields. In other words, $\text{Gal}(L/F) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for $m \in \mathbb{Z}_{>1}$. As such, L is the compositum of the $2^m - 1$ relative quadratic subextensions K_i/F for $1 \leq i \leq 2^m - 1$. Suppose that $(L/F, S, r)$ satisfy hypotheses H_r. Let $v_1, v_2, \ldots, v_r$ be the places that split completely in L. The v_i necessarily split completely in each relative quadratic subextension K_i/F. Finally, as the case $\#S > r+1$ has already been given extensive treatment in the literature [6][20], we restrict our attention to the remaining case where $|S| = r + 1$.

Let $K = K_i$ denote a fixed relative quadratic subextension for which precisely r places of S split in K/F. Let $G = \mathrm{Gal}(K/F) = \langle \tau \rangle$. In the paper of Sands [20, pg. 5], the author uses the method of Rubin [18, Theorem 3.5] to specify the evaluator $\epsilon_{K/F,S}$ for a relative quadratic extension when the set S contains at least $r+2$ primes. We adjust this method to specify the evaluator in our setting. By Theorem 2.3.2, $U_{F,S}/\mu_F$ has rank r as a $\mathbb{Z}$-module. As $\#S_K = 2r+1$, the rank of $U_{K,S}/\mu_K$ is $2r$. Let $u_1, u_2, \ldots, u_r \in K$ form a basis for $U_{K,S}/(U_{F,S}\mu_K)$ modulo its torsion subgroup. The chain of containments $U_{F,S}^2 \subset U_{K,S}^{1+\tau} \subset U_{F,S}$ shows that $U_{K,S}^{1+\tau}$ is of finite index in $U_{F,S}$. Choose $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_r \in K$ such that $\widetilde{u}_1^{1+\tau}, \widetilde{u}_2^{1+\tau}, \ldots, \widetilde{u}_r^{1+\tau}$ form a basis for $U_{K,S}^{1+\tau}/\mu_K^{1+\tau}$ modulo its torsion subgroup. Also let $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_r \in F$ form a $\mathbb{Z}$-basis for $U_{F,S}/\mu_F$. We will need the following lemma.

Lemma 3.2.1. *In $\mathbb{Q}\bigwedge^r U_{L,S}$ one has*

$$\frac{1}{\left[U_{F,S} : U_{K,S}^{1+\tau}\mu_F\right]} \widetilde{u}_1^{1+\tau} \wedge \cdots \wedge \widetilde{u}_r^{1+\tau} = \hat{u}_1 \wedge \cdots \wedge \hat{u}_r.$$

Proof. Let $w = w_F$. Let $x_1^w, x_2^w, \ldots, x_r^w$ generate a full rank submodule of the free $\mathbb{Z}$-module $U_{F,S}^w$. We may then write (writing the units additively) $x_i^w = m_{i1}\hat{u}_1^w + \cdots + m_{ir}\hat{u}_r^w$ for $i = 1, 2, \ldots, r$ to obtain a change of basis matrix with integer entries. The absolute value of the determinant of this matrix is the index of $\langle x_1^w, \ldots, x_r^w \rangle$ in $\langle \hat{u}_1^w, \ldots, \hat{u}_r^w \rangle$ [3, pg. 69]. The map given by $M : U_{F,S}^w \longrightarrow U_{F,S}^w$ induces a $\mathbb{Z}$-module endomorphism on $\bigwedge^r U_{F,S}^w$ which by is given by multiplication by the determinant of M [8, Corollary A2.3(e)]. We therefore have

$$\pm (\det M)\, \hat{u}_1^w \wedge \cdots \wedge \hat{u}_r^w = x_1^w \wedge \ldots x_r^w$$

where we may interchange two x_i if necessary to ensure a $+$ sign. This implies that $(\det M)\, \hat{u}_1 \wedge \cdots \wedge \hat{u}_r = x_1 \wedge \cdots \wedge x_r$ in $\mathbb{Q}\bigwedge^r U_{F,S}$. The proof is completed by observing that $\left[\langle \hat{u}_1, \ldots, \hat{u}_r \rangle : \langle \widetilde{u}_1^{1+\tau}, \ldots, \widetilde{u}_r^{1+\tau} \rangle\right] = \left[U_{F,S} : U_{K,S}^{1+\tau}\mu_F\right]$. □

The evaluator for K/F is given by the sum

$$\epsilon_{K/F} = \frac{h_{F,S}}{2^r w_F \left[U_{F,S} : U_{K,S}^{1+\tau}\mu_F\right]} \widetilde{u}_1^{1+\tau} \wedge \widetilde{u}_2^{1+\tau} \wedge \cdots \wedge \widetilde{u}_r^{1+\tau}$$

$$+ \frac{h_{K,S} w_F}{2^r h_{F,S} w_K Q} u_1^{1-\tau} \wedge u_2^{1-\tau} \wedge \cdots \wedge u_r^{1-\tau}$$

where Q is the order of the torsion subgroup of $U_{K,S}/U_{F,S}\mu_K$. If all $r+1$ primes of S split in the relative quadratic extension K/F we take

$$\epsilon_{K/F} = \frac{h_{F,S}}{2^r w_F \left[U_{F,S} : U_{K,S}^{1+\tau}\mu_F\right]} \widetilde{u}_1^{1+\tau} \wedge \widetilde{u}_2^{1+\tau} \wedge \cdots \wedge \widetilde{u}_r^{1+\tau}. \tag{3.6}$$

One may readily verify the evaluation property in both cases by first applying the trivial character $\chi_0 \in \widehat{G}$ to $\mathcal{R}_{K,S}\left(\epsilon_{K/F}\right)$. One obtains [18, Theorem 3.5]

$$\chi_0\left(\mathcal{R}_{K,S}\left(\epsilon_{K/F}\right)\right) = \frac{h_{F,S}R_{F,S}}{w_F} = \zeta_{F,S}^{(r)}(0). \tag{3.7}$$

Applying the nontrivial character $\chi \in \widehat{G}$ gives

$$\chi\left(\mathcal{R}_{K,S}\left(\epsilon_{K/F}\right)\right) = \frac{h_{K,S} w_F R_{K,S}}{h_{F,S} w_K R_{F,S}} = \frac{\zeta_{K,S}^{(r)}(0)}{\zeta_{F,S}^{(r)}(0)} = L_S^{(r)}(0,\chi). \tag{3.8}$$

It is easy to see that $\epsilon_{K/F} \in \left(\mathbb{Q}\bigwedge^r U_{K,S}\right)_{r,S}$ as there are no $\chi \in \widehat{G}$ for which $r_S(\chi) > r$. Therefore $\epsilon_{K/F}$ is the unique element of $\left(\mathbb{C}\bigwedge^r U_{K,S}\right)_{r,S}$ satisfying these properties.

Sands has shown [20, Proposition 4.5] that the evaluator $\epsilon_{L/F}$ is obtained as follows. Let K_i for $1 \leq i \leq 2^m - 1$ denote the distinct relative quadratic subextensions K_i/F of L/F and ϵ_i the evaluator $\epsilon_{K_i/F}$. Then

$$\epsilon_{L/F} = \frac{1}{\left(2^{m-1}\right)^r}\left[-\left(2^m - 2\right)2^{-r}\epsilon_{F/F} + \left(\epsilon_1 + \cdots + \epsilon_{2^m-1}\right)\right]. \tag{3.9}$$

where

$$\epsilon_{F/F} = \frac{h_{F,S}}{w_F} \hat{u}_1 \wedge \hat{u}_2 \cdots \wedge \hat{u}_r. \tag{3.10}$$

Suppose now that $u_{1,i}, u_{2,i}, \ldots u_{r,i} \in K_i$ form a basis for $U_{K_i,S}/U_{F,S}\mu_{K_i}$ modulo its torsion subgroup. Similarly let $\widetilde{u}_{1,i}, \widetilde{u}_{2,i}, \ldots, \widetilde{u}_{r,i} \in K_i$ such that $\widetilde{u}_{1,i}^{1+\tau_i}, \widetilde{u}_{2,i}^{1+\tau_i}, \ldots, \widetilde{u}_{r,i}^{1+\tau_i}$ form a basis for $U_{K_i,S}^{1+\tau_i}/\mu_{K_i}^{1+\tau_i}$. We may rewrite $\epsilon_{L/F}$ as

$$\begin{aligned} 2^{mr-r}\epsilon_{L/F} = \quad & \sum_{i=1}^{2^m-1} \left(\frac{h_{F,S}}{2^r w_F \left[U_{F,S} : U_{K_i,S}^{1+\tau_i}\mu_F\right]} \widetilde{u}_{1,i}^{1+\tau_i} \wedge \cdots \wedge \widetilde{u}_{r,i}^{1+\tau_i} - \frac{h_{F,S}}{2^r w_F} \hat{u}_1 \wedge \cdots \wedge \hat{u}_r \right) \\ & + \sum_{\#S_{K_i}=2r+1} \left(\frac{h_{K_i,S} w_F}{2^r h_{F,S} w_{K_i} Q_i} u_{1,i}^{1-\tau_i} \wedge \cdots \wedge u_{r,i}^{1-\tau_i} \right) + \frac{1}{2^r}\epsilon_{F/F} \end{aligned}$$

We note that the condition that $\#S_{K_i} = 2r+1$ is equivalent to specifying that exactly r primes of S split in K_i/F. For all i, the set $\{\widetilde{u}_{1,i}^{1+\tau_i}, \ldots \widetilde{u}_{r,i}^{1+\tau_i}\}$ generates a $\mathbb{Z}$-submodule of that generated by the set $\{\hat{u}_1, \ldots, \hat{u}_r\}$ of index $\left[U_{F,S} : U_{K_i,S}^{1+\tau_i}\mu_F\right]$. Applying Lemma 3.2.1 for each i, one has

$$\frac{h_{F,S}}{2^r w_F \left[U_{F,S} : U_{K_i,S}^{1+\tau_i}\mu_F\right]} \widetilde{u}_{1,i}^{1+\tau_i} \wedge \cdots \wedge \widetilde{u}_{r,i}^{1+\tau_i} = \frac{h_{F,S}}{2^r w_F} \hat{u}_1 \wedge \cdots \wedge \hat{u}_r.$$

The first sum vanishes leaving us with

$$\epsilon_{L/F} = \quad \frac{1}{2^{mr-r}} \sum_{i=1}^{2^m-1} \left(\frac{h_{K_i,S} w_F}{2^r h_{F,S} w_{K_i} Q_i} u_{1,i}^{1-\tau_i} \wedge \cdots \wedge u_{r,i}^{1-\tau_i} \right) + \frac{1}{2^{mr}}\epsilon_{F/F}. \tag{3.11}$$

Our goal is to find conditions under which the evaluator $\epsilon_{L/F}$ lies in $\bigwedge_1^r U_{L,S}$ (Definition 3.1.8). We will need the following lemma. Let $G = \mathrm{Gal}(L/F)$ and let $N_G = \sum_{g \in G} g$ denote the *norm element* of $\mathbb{Z}[G]$.

Lemma 3.2.2. *Let* $\phi \in U_{L,S}^*$ *and* $u^k \in U_{F,S}$ *for some integer* k. *If* $\sigma \in G$ *then one has*

$$\sigma \cdot \phi(u) = nN_G \quad \textit{for some} \quad n \in \mathbb{Z}.$$

Proof. This is a straightforward computation. Let $\phi(u) = \sum_{g \in G} n_g g$ and $n = n_{\mathrm{id}}$. Since $(u^k)^\sigma = u^k$, one has

$$\phi((u^k)^\sigma) = k\sigma \cdot \phi(u) = k\sigma \cdot \sum_{g \in G} n_g g = k \sum_{g \in G} n_g (g\sigma) = k\phi(u) = k \sum_{g \in G} n_g g. \tag{3.12}$$

Therefore, $n_{g\sigma} = n_g$ for all $\sigma \in G$. Therefore $\phi(u) = nN_G$. □

3.3 Theorems

Based on this computation and the definition of the lattice $\bigwedge_1 U_{L,S}$ we are ready to state a sufficient condition for $\frac{1}{2^{mr}}\epsilon_{F/F}$ to lie in $\bigwedge_1 U_{L,S}$.

Theorem 3.3.1. *If* $2^m \mid h_{F,S}$ *then one has* $\frac{1}{2^{mr}}\epsilon_{F/F} \in \bigwedge_1 U_{L,S}$.

Proof. Let $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_r$ give a basis for $U_{F,S}/\mu_F$. Let $\phi_1, \phi_2, \ldots, \phi_{r-1} \in U_{L,S}^*$. We need to show that

$$\phi_1 \wedge \cdots \wedge \phi_{r-1} \left(\epsilon_{F/F}\right) = \frac{h_{F,S}}{w_F 2^{mr}} \sum_{k=1}^{r} (-1)^k \left(\det_{j \neq k} \phi_i(\hat{u}_j)\right) \cdot u_k \in \frac{1}{w_L} \widetilde{U_{L,S}}.$$

Since $w_F \mid w_L$ it suffices to show that for all $k = 1, \ldots, r$ we have

$$\frac{h_{F,S}}{2^{mr}} \left(\det_{j \neq k} \phi_i(\hat{u}_j)\right) \cdot \hat{u}_k \in \widetilde{U_{L,S}}.$$

From the lemma we know that

$$\phi_i(\hat{u}_j) = n_{ij} N_G \quad \forall \quad i, j. \tag{3.13}$$

This implies that

$$\det_{j \neq k} \phi_i(\hat{u}_j) = \det(n_{ij}) N_G^{r-1}. \tag{3.14}$$

Since $\hat{u}_k \in F$, we have

$$\begin{aligned} \frac{h_{F,S}}{2^{mr}} \left(\det_{j \neq k} \phi_i(\hat{u}_j) \right) \cdot \hat{u}_k &= \frac{h_{F,S}}{2^{mr}} \det(n_{ij}) N_G^{r-1} \cdot \hat{u}_k \\ &= \frac{h_{F,S}}{2^{mr}} \det(n_{ij}) (2^m)^{r-1} \cdot \hat{u}_k \\ &= \frac{h_{F,S}}{2^m} \det(n_{ij}) \cdot \hat{u}_k. \end{aligned}$$

As the factor $\det(n_{ij})$ is clearly an integer the proof is complete. □

Theorem 2.1 of [20] shows that $\epsilon_{L/F} - \frac{1}{2^{mr}} \epsilon_{F/F}$ lies in $\bigwedge_1 U_{L,S}$ whenever L/F is multiquadratic. Combining this with the previous result gives us the following result.

Corollary 3.3.2. *If $G = \mathrm{Gal}(L/F)$ has exponent 2 and $2^m \mid h_{F,S}$ then conjecture $C'(L/F, S, r)$ holds.*

In order to prove the full conjecture we will need stronger hypotheses. Let $\mathrm{rk}_2(Cl_{F,S})$ denote the 2-rank of the S-class group of F. Sands has shown [20, Theorem 2.2] that the element $\epsilon_{L/F} - \frac{1}{2^{mr}} \epsilon_{F/F} \in \bigwedge_0^r U_{L,S}$ provided that $\mathrm{rk}_2(Cl_{F,S}) \geq m$. Note that this is stronger than the assumption that $2^m \mid h_{F,S}$.

Theorem 3.3.3. *If $2^m \mid h_{F,S}$ then one has $\frac{1}{2^{mr}} \epsilon_{F/F} \in \bigwedge_0^r U_{L,S}$.*

Proof. Let $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_r$ give a basis for $U_{F,S}/\mu_F$. From the proof of Theorem 3.3.1 it suffices to show that for $k = 1, \ldots, r$ we have $\frac{w_L}{w_F} \cdot \hat{u}_k \in \widetilde{U_{L,S}^{\mathrm{ab}}}$. This element is the image in $\widetilde{U_{L,S}}$ of $u_k^{\frac{w_L}{w_F}}$. The field

$$L\left(\left(u_k^{\frac{w_L}{w_F}} \right)^{\frac{1}{w_L}} \right) = L\left(u_k^{\frac{1}{w_F}} \right) \tag{3.15}$$

is the compositum of the fields L and $F\left(u_k^{\frac{1}{w_F}}\right)$, both of which are abelian over F. This proves that $\frac{1}{2^{mr}}\epsilon_{F/F} \in \bigwedge_0 U_{L,S}$. $\square$

Combining our result with the work of Sands we have the

Corollary 3.3.4. *If $\#S \geq r+1$ and G has exponent 2 and order 2^m then Conjecture $C(L/F,S,r)$ holds provided that $\#S + rk_2(Cl_{F,S}) \geq r+m+1$ and, when $\#S + \mathrm{rk}_2(Cl_{F,S}) \leq r+m+2$, we have $\sqrt{-1} \notin L$.*

We have noticed that the truth of the conjectures in the multiquadratic case depends on the exact power of 2 that divides the S-class number $h_{F,S}$ of S. The following lemma establishes a lower bound for this power of 2.

Lemma 3.3.5. *If $S = \{v_1, v_2, \ldots, v_r, v_{r+1}\}$ where v_{r+1} is a finite prime that does not split in L/F and divides $(p) \subset \mathbb{Z}$, where $p \neq 2$, then $2^{m-2} \mid h_{F,S}$. In fact, we show that $rk_2(Cl_{F,S}) \geq m-2$.*

Proof. Let $w_{r+1} = \mathfrak{P}$ lie over $v_{r+1} = \mathfrak{p}$. Let $G_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ be the decomposition and the inertia groups associated to $\mathfrak{P} \mid \mathfrak{p}$ respectively 2.5.1. Let

$$G_{-1} \supset G_0 = I_{\mathfrak{P}} \supset G_1 \supset \cdots \tag{3.16}$$

be the chain of higher ramification groups associated to $\mathfrak{P} \mid \mathfrak{p}$. We know that $I_{\mathfrak{P}}/G_1 \cong H \leq (\mathbb{F}_{p^n})^{\times}$ [13, Proposition II.10.2]. Also, G_1 is the unique p-Sylow subgroup of $I_{\mathfrak{P}}$ [13]. Therefore $G_1 = 1$ and we have $I_{\mathfrak{P}} \cong H \leq (\mathbb{F}_{p^n})^{\times}$. This implies that either $\#I_{\mathfrak{P}} = 1$ or 2. Let $H_{F,S}$ be the maximal abelian extension of F in which all primes of S split completely. If $L^{G_{\mathfrak{P}}}$ denotes the field fixed by $G_{\mathfrak{P}}$ then $L^{G_{\mathfrak{P}}} \subset H_{F,S}$. As $[L : L^{I_{\mathfrak{P}}}] \leq 2$ and $L^{I_{\mathfrak{P}}}/L^{G_{\mathfrak{P}}}$ is a cyclic extension one knows that $[L^{G_{\mathfrak{P}}} : F] \geq 2^{m-2}$. It follows that $rk_2(Cl_{F,S}) \geq m-2$ and so $2^{m-2} \mid h_{F,S}$. $\square$

3.4 An Example

We provide an example for which Theorem 3.3.3 does not apply. Let $F = \mathbb{Q}(\sqrt{165})$ and $L = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{11})$. One has $G = \mathrm{Gal}(L/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $S = \{\infty_1, \infty_2, 2\mathcal{O}_F\}$. The two infinite primes split completely in L/F while the ideal generated by 2 (this ideal is prime in $F/\mathbb{Q}$) ramifies. Therefore $r = 2$ is the order of vanishing of the Artin L-functions associated to G. The quotient $U_{F,S}/\mu_{F,S}$ is generated by $\frac{13+\sqrt{165}}{2}$ and 2. The S-class number of F is $h_{F,S} = 2$ and $w_F = w_L = 2$.

We would like to determine whether the element $\frac{1}{2^{mr}}\epsilon_{F/F} = \frac{h_{F,S}}{w_F 2^4}\frac{13+\sqrt{165}}{2} \wedge 2$ lies in $\bigwedge_1^r U_{L,S}$. In order to do this we must find a basis for $U_{L,S}^*$. One has a canonical isomorphism [18, page 37]

$$\mathrm{Hom}_{\mathbb{Z}}(U_{L,S}, \mathbb{Z}) \xrightarrow{\Phi} \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S}, \mathbb{Z}[G]) \tag{3.17}$$

$$\phi \longmapsto (u \mapsto \sum_{\sigma \in G} \phi(u^\sigma) \cdot \sigma^{-1}). \tag{3.18}$$

We compute a basis $u_1, u_2, \ldots, u_9$ for $U_{L,S}/\mu_L$ using Sage [19]. This basis determines a dual basis $u_1^*, u_2^*, \ldots, u_9^*$ for $\mathrm{Hom}_{\mathbb{Z}}(U_{L,S}/\mu_L, \mathbb{Z})$. We now represent $\frac{13+\sqrt{165}}{2}$ and 2 in terms of the basis for $U_{L,S}/\mu_K$ and obtain

$$\frac{13+\sqrt{165}}{2} = u_4^{-2}$$

$$2 = -u_1^2 u_2 u_3^{-1} u_8^2 u_9^2.$$

Let ϕ_i denote the image of u_i in $\mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S}, \mathbb{Z}[G])$ under the composition of the map sending u_i to u_i^* and Φ. The homomorphism in 3.17 above gives $\phi_i(u_j) = \delta_{ij}$. In particular we apply ϕ_2 to $\frac{h_{F,S}}{w_F 2^4}\frac{13+\sqrt{165}}{2} \wedge 2$ to obtain in $\mathbb{Q}U_{L,S}$

$$\phi_2\left(\frac{h_{F,S}}{w_F 2^4}\frac{13+\sqrt{165}}{2}\wedge 2\right) = \frac{1}{16}\left((-1)\phi_2\left(\frac{13+\sqrt{165}}{2}\right)\cdot 2 + \phi_2(2)\cdot\left(\frac{13+\sqrt{165}}{2}\right)\right)$$

$$= \frac{1}{16}\left(0\cdot 2 + N_G\cdot\left(\frac{13+\sqrt{165}}{2}\right)\right).$$

In $U_{L,S}$, this element is

$$\left(\frac{13+\sqrt{165}}{2}\right)^{\frac{N_G}{16}} = \left(\frac{13+\sqrt{165}}{2}\right)^{\frac{1}{4}}$$

which (as $\frac{13+\sqrt{165}}{2}$ is a square in $U_{L,S}$) maps to an element of $\frac{1}{w_L}\widetilde{U_{L,S}}$. One can check in a similar way that $\phi_i(u_j) \in \frac{1}{w_L}\widetilde{U_{L,S}}$ for $i, j = 1, \ldots, 9$. As any element of $U^*_{L,S}$ may be expressed as a linear combination of the ϕ_i we see that $\frac{1}{2^{mr}}\epsilon_{F/F} = \frac{h_{F,S}}{w_F 2^4}\frac{13+\sqrt{165}}{2}\wedge 2 \in \bigwedge_1^r U_{L,S}$ yet 2^m does not divide $h_{F,S}$. The previous example suggests that when a fundamental unit of the base field is a square of an S-unit in the top field, the power of 2 required to divide $h_{F,S}$ decreases by one. This leads us to the following result.

Proposition 3.4.1. *If $u_1, u_2, \ldots, u_r$ form a basis for $U_{F,S}/\mu_{F,S}$ and $u_i = v^2$ for some $v \in U_{L,S}$ and $i \in \{1, 2, \ldots r\}$ then conjecture $C'(L/F, S, r)$ holds provided $2^{m-1} \mid h_{F,S}$*

Proof. We return to the computation performed in the proof of Theorem 3.3.1. We wish to show that for each $k = 1, \ldots, r$ we have

$$\frac{h_{F,S}}{2^{mr}}\left(\det_{j\neq k}\phi_i(\hat{u}_j)\right)\cdot\hat{u}_k \in \widetilde{U_{L,S}}.$$

If $k = 1$ we have

$$\frac{h_{F,S}}{2^{mr}} \left(\det_{j \neq 1} \phi_i(\hat{u}_j) \right) \cdot \hat{u}_1 = \frac{h_{F,S}}{2^{mr}} \det(n_{ij}) N_G^{r-1} \cdot v^2$$

$$= \frac{h_{F,S}}{2^{mr}} \det(n_{ij})(2^m)^{r-1} \cdot v^2 = \frac{h_{F,S}}{2^{m-1}} \det(n_{ij}) \cdot v.$$

This element lies in $\widetilde{U_{L,S}}$ provided $2^{m-1} \mid h_{F,S}$. Now let us suppose that $k \neq 1$. Then we have

$$\frac{h_{F,S}}{2^{mr}} \left(\det_{j \neq k} \phi_i(\hat{u}_j) \right) \cdot \hat{u}_k = \frac{h_{F,S}}{2^{mr}} \det \begin{pmatrix} \phi_1(u_1) & \dots & \phi_1(u_r) \\ \vdots & \ddots & \vdots \\ \phi_r(u_1) & \dots & \phi_r(u_r) \end{pmatrix} \cdot u_k$$

$$= \frac{h_{F,S}}{2^{mr}} \det \begin{pmatrix} \phi_1(v^2) & \dots & \phi_1(u_r) \\ \vdots & \ddots & \vdots \\ \phi_r(v^2) & \dots & \phi_r(u_r) \end{pmatrix} \cdot u_k$$

$$= \frac{h_{F,S}}{2^{mr}} \det \begin{pmatrix} 2\phi_1(v) & \dots & \phi_1(u_r) \\ \vdots & \ddots & \vdots \\ 2\phi_r(v) & \dots & \phi_r(u_r) \end{pmatrix} \cdot u_k$$

$$= \frac{h_{F,S}}{2^{mr}} 2 \det(n_{ij}) N_G^{r-1} \cdot u_k$$

$$= \frac{h_{F,S}}{2^{m-1}} det(n_{ij}) \cdot u_k.$$

This lies in $\widetilde{U_{L,S}}$ provided $2^{m-1} \mid h_{F,S}$. □

Chapter 4

New Results, Computations, and Future Directions

4.1 Statement of Theorems

In this chapter we will prove three theorems that will provide us with an infinite collection of settings in which we are able to prove Popescu's Conjecture. For the rest of the chapter p_i, q_j, and l will denote rational primes for all natural numbers i and j. If Q is a subset of $\mathbb{Z}$ then $F(\sqrt{Q})$ will denote the field obtained by adjoining the square root of each element of Q to the field F. The first two theorems will focus on a real quadratic base field F. We will denote the two infinite primes of F by ∞_1 and ∞_2 and its fundamental unit by η.

Theorem 4.1.1. *Suppose $p_1 p_2 \cdots p_t = m \equiv 5 \pmod 8$. Let $Q = \{q_1, q_2, \ldots, q_k\} \subsetneq \{p_1, p_2, \ldots, p_t\}$ for which at least one $q_i \not\equiv 1 \pmod 4$. Let $F = \mathbb{Q}(\sqrt{m}), L = F(\sqrt{Q})$, and $S = \{\infty_1, \infty_2, 2\mathcal{O}_F\}$.*

1. *If $\#Q < t - 1$, then conjecture $C(L/F, S, 2)$ holds.*

2. *If $\#Q = t - 1$ and $\eta = u^2$ for $u \in U_{L,S}$, then conjecture $C'(L/F, S, 2)$ holds.*

Theorem 4.1.2. *Suppose* $p_1 \equiv p_2 \equiv \cdots \equiv p_{2t+1} \equiv 5 \pmod 8$. *Let* $Q \subsetneq \{p_1, p_2, \ldots, p_{2t+1}\}$. *Let* $l\mathcal{O}_F$ *be a prime ideal. Let* $F = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_{2t+1}}), L = F(\sqrt{Q \cup \{l\}})$, *and* $S = \{\infty_1\infty_2, l\mathcal{O}_F\}$.

1. *If* $\#Q < 2t$, *then conjecture* $C(L/F, S, 2)$ *holds.*

2. *If* $\#Q = 2t$, *then conjecture* $C'(L/F, S, 2)$ *holds.*

In the next theorem F will be a purely real quartic field with infinite primes ∞_1, ∞_2, and ∞_3.

Theorem 4.1.3. *Suppose* $p_1 \equiv p_2 \equiv \cdots \equiv p_t \equiv 5 \pmod 8$. *Let* $Q \subset \{p_1, p_2, \ldots, p_t\}$. *Let* $l\mathcal{O}_F$ *be a prime ideal. Let* $F = \mathbb{Q}(\sqrt[4]{2p_1^2 p_2^2 \cdots p_t^2}), L = F(\sqrt{Q \cup \{l\}})$, *and* $S = \{\infty_1\infty_2, \infty_3, l\mathcal{O}_F\}$.

1. *If* $\#Q < t$, *then conjecture* $C(L/F, S, 3)$ *holds.*

2. *If* $\#Q = t$, *then conjecture* $C'(L/F, S, 3)$ *holds.*

4.2 Quadratic Base Fields

The example of section 2.3 leads us to consider the following class of examples. Let $F = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_t})$ where $p_1, p_2, \ldots, p_t$ are rational primes for which the product $p_1 p_2 \cdots p_t \equiv 5 \pmod 8$. Let $Q = \{q_1, q_2, \ldots, q_k\} \subsetneq \{p_1, p_2, \ldots, p_t\}$ be a set of primes for which at least one $q_i \not\equiv 1 \pmod 4$. Let $L = F(\sqrt{Q})$. The extension L/F is multiquadratic with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^k$. The ideal $2\mathcal{O}_F$ is prime and ramifies in each relative quadratic extension $F(\sqrt{q_i})/F$ for which $q_i \not\equiv 1 \pmod 4$. All other prime ideals of $\mathcal{O}_F$ are unramified in L/F. Genus Theory [12, Proposition 2.12] shows that we have $\mathrm{rk}_2(Cl_F) = t - 2$. If we let $S = \{\infty_1, \infty_2, 2\mathcal{O}_F\}$, the data $(L/F, S, 2)$ satisfies Hypotheses 3.1.1.

As an example let $F = \mathbb{Q}(\sqrt{3 \cdot 5 \cdot 7 \cdot 13}) = \mathbb{Q}(\sqrt{1365})$. Since the ideal $2\mathcal{O}_F$ is principal, we have $\mathrm{rk}_2(Cl_{F,S}) = \mathrm{rk}_2(Cl_F) = 2$. For the five biquadratic extensions $L = F(\sqrt{3}, \sqrt{5})$, $F(\sqrt{3}, \sqrt{7})$, $F(\sqrt{3}, \sqrt{13})$, $F(\sqrt{5}, \sqrt{7})$, and $F(\sqrt{7}, \sqrt{13})$ the data $(L/F, S, 2)$ satisfies the hypotheses of Corollary 3.3.4, proving conjecture $C(L/F, S, 2)$.

When $L = F(\sqrt{3}, \sqrt{5}, \sqrt{7})$ the extension L/F has degree $n = 2^3$ and hence does not satisfy the hypotheses of Corollary 3.3.4. We use the following routine in Magma [1] to determine if the fundamental unit of F, η, becomes a square in $U_{L,S}$.

Routine 4.2.1.

```
P<x>:=PolynomialRing(Rationals());

F<t>:=NumberField(x^2 - 3*5*7*13);
L<u,v,w>:=ext<F|x^2 - 3, x^2 - 5, x^2 - 7>;
Labs:=AbsoluteField(L);
O_F:=MaximalOrder(F);
U_F,m_F:=UnitGroup(O_F);
u:=m_F(U_F.2);
IsSquare(L!u)
```

The commands initialize the fields F and L as well as the integers and units of F. The final command asks whether or not η is a square in L. The output is `yes`. As η is a square in L, it is a square in $U_{L,S}$. We apply proposition 3.4.1 to conclude that conjecture $C'(L/F, S, 2)$ holds.

To find a larger collection of new settings of Popescu's conjecture take $F = \mathbb{Q}(\sqrt{3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot 41})$. We have $\mathrm{rk}_2(Cl_{F,S}) = 5$. By adjoining square roots of primes in the set $\{3, 5, 7, 11, 17, 23, 41\}$ to F we are able to find $18, 34, 35, 21$ new settings with $G = (\mathbb{Z}/2\mathbb{Z})^s$ with $s = 2, 3, 4, 5$, respectively. We are able to implement routine

4.2.1 to see that when we take $L = F(\sqrt{3}, \sqrt{5}, \ldots, \sqrt{23})$, η becomes a square in L. We are therefore able to prove conjecture $C'(L/F, S, 2)$ for this extension L/F.

Proof of Theorem 4.1.1. Genus theory shows [12, Proposition 2.12] that the 2-rank of the class group of F has $\mathrm{rk}_2(Cl_F) = t - 2$. From definition 2.3.3 and our choice of S we see that $\mathrm{rk}_2(Cl_{F,S}) = \mathrm{rk}_2(Cl_F)$. If $\#Q \leq t - 2$ then $[L : F] \leq 2^{t-2}$ and the conditions of Corollary 3.3.4 are satisfied. Therefore conjecture $C(L/F, S, 2)$ holds.

Now suppose that $\#Q = t - 1$ and $\eta = u^2$ for $u \in U_{L,S}$. Then $[L : F] = 2^{t-1}$ and the conditions of proposition 3.4.1 are satisfied. Conjecture $C'(L/F, S, 2)$ follows. □

This infinite class of examples features the same set of primes $S = \{\infty_1, \infty_2, 2\mathcal{O}_F\}$ of the base field. We would like to consider extensions L/F where a prime $\mathfrak{p}$ not dividing 2 ramifies. To accomplish this we choose a set of primes $P = \{p_1, p_2, \ldots p_{2t+1}\}$ for which

$$p_1 \equiv p_2 \equiv \cdots \equiv p_{2t+1} \equiv 5 \pmod 8.$$

Let $F = \mathbb{Q}(\sqrt{p_1 p_2 \cdots p_{2t+1}})$. The extension $F/\mathbb{Q}$ is unramified away from $p_1, \ldots p_{2t+1}$. By the Chebotarev Density Theorem [13, Theorem VII.13.4] there exist infinitely many primes l of $\mathbb{Z}$ that are inert in $F/\mathbb{Q}$. Choose one such l. Let $Q = \{q_1, q_2, \ldots, q_k\} \subsetneq P$ and take $L = F\left(\sqrt{Q \cup \{l\}}\right)$. The extension L/F is multiquadratic with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^{k+1}$. Let $S = \{\infty_1, \infty_2, l\mathcal{O}_F\}$. Since the two infinite places split in the extension L/F, the data $(L/F, S, 2)$ satisfies Hypotheses 3.1.1.

As an example let us consider $F = \mathbb{Q}(\sqrt{5 \cdot 13 \cdot 29}) = \mathbb{Q}(\sqrt{1885})$. We let $l = 37$ and observe that the ideal $37\mathcal{O}_F$ is prime. Let $S = \{\infty_1, \infty_2, 37\mathcal{O}_F\}$. Since 1885 may be written as the sum of two squares, Genus Theory [12, Proposition 2.12] shows that $\mathrm{rk}_2(Cl_F) = 2$ For the 3 biquadratic extensions $L = F(\sqrt{5}, \sqrt{37}), F(\sqrt{13}, \sqrt{37})$, and $F(\sqrt{29}, \sqrt{37})$ the data $(L/F, S, 2)$ satisfies the hypotheses of Corollary 3.3.4, proving conjecture $C(L/F, S, 2)$.

When $L = F(\sqrt{5}, \sqrt{13}, \sqrt{37})$ the extension L/F has degree $n = 2^3$ and therefore fails to satisfy the hypotheses of Corollary 3.3.4. The fundamental S-unit 37 is a square in $U_{L,S}$ as $\sqrt{37} \in L$. We may therefore apply proposition 3.4.1 to see that conjecture $C'(L/F, S, 2)$ holds.

Proof of Theorem 4.1.2. Genus theory [12, Theorem 2.12] shows that one has $\mathrm{rk}_2(Cl_F) = 2t$. Since $l\mathcal{O}_F$ is principal we have $\mathrm{rk}_2(Cl_{F,S}) = \mathrm{rk}_2(Cl_F)$. If $\#Q < 2t$ then $[L : F] \leq 2^{2t}$ and the conditions of Corollary 3.3.4 are satisfied. Conjecture $C(L/F, S, 2)$ follows.

Now suppose that $\#Q = 2t$. Then $[L : F] = 2^{2t+1}$ and the conditions of proposition 3.4.1 are satisfied as the fundamental S-unit l is a square in $U_{L,S}$. Therefore Conjecture $C'(L/F, S, 2)$ is true. □

4.3 Quartic Base Fields

We would like to find a new collection of base fields F that will allow us to construct new settings for Popescu's conjecture. A paper [15] of Parry led us specifically to the following setting. Let $P = \{p_1, p_2, \ldots, p_t\}$ be a set of rational primes satisfying

$$p_1 \equiv p_2 \equiv \cdots \equiv p_t \equiv 5 \pmod 8.$$

Let $F = \mathbb{Q}\left(\sqrt[4]{2p_1^2 p_2^2 \cdots p_t^2}\right)$.

Proposition 4.3.1. *There exists a rational prime l for which the ideal $l\mathcal{O}_F$ is prime.*

Proof. Let $E = F(i)$. One observes that $G = \mathrm{Gal}(E/\mathbb{Q}) \cong D_8$. Let H be the subgroup of G fixing F. Let $\sigma \in G$ be an element of order 4. By the Chebotarev Density Theorem [13, VII.13.4], we may produce a prime $\mathfrak{P}_l$ of E dividing l of $\mathbb{Q}$ for which $\mathfrak{P}_l \mid l$ is unramified and $\sigma = \sigma_{\mathfrak{P}_l}$ is the Frobenius automorphism of $\mathfrak{P}_l \mid l$ (the theorem guarantees that such primes have positive density). Now consider the coset decomposition

$$G = H \cup H\sigma \cup H\sigma^2 \cup H\sigma^3.$$

The cosets above also give us a cycle of σ of length 4. By the rules for prime decomposition in non-Galois extensions [11, III.2.7] l is inert in $F/\mathbb{Q}$. Therefore the ideal $l\mathcal{O}_F$ is prime. □

Let l be a rational prime for which $l\mathcal{O}_F$ is a prime ideal. Let $Q = \{q_1, q_2, \ldots, q_k\} \subsetneq P$ and take $L = F(\sqrt{Q \cup \{l\}})$. The extension L/F is multiquadratic, with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^{k+1}$. The ideal $l\mathcal{O}_F$ ramifies in L/F while all other prime ideals of $\mathcal{O}_F$ are unramified. The three infinite primes, $\{\infty_1, \infty_2, \infty_3\}$, of F split completely in L/F. If we let $S = \{\infty_1, \infty_2, \infty_3, l\mathcal{O}_F\}$ the data $(L/F, S, 3)$ satisfy hypotheses 3.1.1.

As an example, let $F = \mathbb{Q}(\sqrt[4]{2 \cdot 5^2 \cdot 13^2 \cdot 29^2})$. We observe that $\mathrm{rk}_2(Cl_F) = 2$. The ideal $37\mathcal{O}_F$ is prime so we let $S = \{\infty_1, \infty_2, \infty_3, 37\mathcal{O}_F\}$. For the 4 biquadratic extensions $L = F(\sqrt{5}, \sqrt{37}), F(\sqrt{13}, \sqrt{37})$, and $F(\sqrt{29}, \sqrt{37})$ the data $(L/F, S, 3)$ satisfies the hypotheses of Corollary 3.3.4, proving conjecture $C(L/F, S, 3)$. When $L = F(\sqrt{5}, \sqrt{13}, \sqrt{37})$ the extension L/F has degree $n = 2^3$, one power of 2 too large to apply Corollary 3.3.4. However, the fundamental S-unit, 37, becomes a square in $U_{L,S}$. We are therefore able to apply proposition 3.4.1 to prove Conjecture $C'(L/F, S, 3)$.

Proof. We observe that for each odd prime p_i dividing the discriminant of $F/\mathbb{Q}$ the extension $F(\sqrt{p_i})/F$ is unramified. The only prime $\mathfrak{p}_i \subset \mathcal{O}_F$ that could ramify in this extension would divide p_i. Observe that $F(\sqrt{p_i})/F = F\left(\sqrt{\sqrt{2}p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_t}\right)/F$ and $\mathfrak{p}_i$ could not possibly ramify in the second extension. The unique prime $\mathfrak{p}_2$ of F dividing 2 does not ramify in $F(\sqrt{p_i})$ as $p_i \equiv 5 \pmod 8$. Therefore $F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_t})$ is contained in the Hilbert class field of F. One therefore has $\mathrm{rk}_2(Cl_F) \geq t$. Our choice of S gives $\mathrm{rk}_2(Cl_F) = \mathrm{rk}_2(Cl_{F,S})$.

Suppose that $\#Q < t$. Then $[L : F] \leq 2^t$. The conditions of Corollary 3.3.4 are therefore satisfied and Conjecture $C(L/F, S, 3)$ is true.

Now suppose that $\#Q = t$. Then $[L : F] = 2^{t+1}$ and the conditions of proposition 3.4.1 are satisfied as the fundamental S-unit l is a square in $U_{L,S}$. Therefore Conjecture $C'(L/F, S, 3)$ is true.

□

4.4 Future Directions

We intend to work on strengthening Corollary 3.3.4 in order to prove conjecture $C(L/F, S, r)$ for all multiquadratic conjectures under less restrictive assumptions. This was half of the original goal of this research. The other half was extending the known results in the case where $\#S = r + 1$. Our investigations with real quadratic base fields leads us to make the following conjecture.

Conjecture 4.4.1. *Let F, S, Q, and η be as described in Theorem 4.1.1. If $\#Q = t - 1$ then $\eta \in U^2_{L,S}$.*

The extension L/F is the largest unramified abelian extension of F whose Galois group has exponent 2. The condition that the fundamental unit becomes a square in L is satisfied provided that $F(\sqrt{\eta})/F$ is unramified. The only prime ideal that could possibly ramify in this extension is $2\mathcal{O}_F$. By a theorem of Hecke [4, Theorem 10.2.9] $2\mathcal{O}_F$ ramifies in L/F provided η is not a square $\pmod 4$. We have found that Conjecture 4.4.1 is true in every setting we have encountered by implementing Routine 4.2.1.

We are currently in the process of writing an algorithm that will verify Conjecture 4.4.1 for a large number of examples. Routine 4.2.1 requires that one input a set of primes $\{p_1, p_2, \ldots p_t\}$ that is selected to satisfy the hypotheses of Theorem 4.1.1. The desired algorithm will accept a pair of natural numbers t and M. It would then verify the conjecture for all base fields $\mathbb{Q}(\sqrt{m})$ where m is the product of t primes less than or equal to M that satisfy the hypotheses of Theorem 4.1.1.

Non-abelian and p-adic L-functions are interesting settings that are currently being explored. We would like to formulate and prove a non-abelian version of Popescu's conjecture. The work of Burns and his collaborators on the Equivariant Tamagawa Number Conjecture has given number theorists a new way of considering higher order integral Stark type conjectures. There is much work to be done in unraveling what this work means for Artin L-functions attached to non-abelian extensions in the cases when the set of primes S of the base field fails to kill the class group $Cl_{F,S}$.

A natural question to ask is whether one may make similar progress in proving the conjectures when $\mathrm{Gal}(L/F) \cong (\mathbb{Z}/p\mathbb{Z})^m$ for $p > 2$. Caleb Emmons considers this question for $p = 3$ briefly in his doctoral dissertation [7]. The primary difficulty with these sorts of extension is that there is currently no known evaluator element for the general relative cubic extension as there is for the general relative quadratic extension. Upon formulating an evaluator element one would seek to build up the full evaluator $\epsilon_{L/F}$ from the evaluators for the cubic subextensions.

Bibliography

[1] Wieb Bosma, John Cannon, and Catherine Playous, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24(3-4) 1997, 235–265.

[2] D. Burns, *Congruences Between Derivatives of Abelian L-functions at* $s = 0$, Invent. Math. **169** (2007), 451–499

[3] H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics **138**, Vol. 138, Springer, New York, 1993.

[4] H. Cohen, *Advanced Topics in Computational Number Theory*, Graduate Texts in Mathematics **193**, Vol. 193, Springer, New York, 2000.

[5] D. S. Dummit and R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc. , Hoboken, NJ, 2004.

[6] D. S. Dummit, J.W. Sands, B. Tangedal *Stark's Conjecture in Multi-quadratic Extensions, Revisited*, J. Theor. Nombres Bordeaux, 15(1) (2001), 83-97.

[7] C. Emmons, *Higher Order Integral Stark-Type Conjectures*, Ph. D Thesis,University of California, San Diego, 2006.

[8] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 2004.

[9] B.H. Gross, *On the Values of Abelian L-functions at* $s = 0$, J. Fac. Sci. Univ. Tokyo Sect. Math. 35(1) 1988, 177–197.

[10] A. Hayward, *Congruences Satisfied by Stark Units*, Ph. D Thesis,King's College, London, 2004.

[11] G. Janusz, *Algebraic Number Fields*, Volume 7 of *Graduate Studies in Mathematics*, Amer. Math. Soc. , Providence, 1996

[12] F. Lemmermeyer, *Reciprocity Laws: From Euler to Eisenstein*, Springer-Verlag, Berlin, 2000.

[13] J. Neukirch, *Algebraic Number Theory*, Volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Priciples of Mathematical Sciences]*, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[14] PARI/GP, version `2.3.0`, Bordeaux, 2006, http://pari.math.u-bordeaux.fr/ .

[15] C.J. Parry, *A Genus Theory for Quartic Fields*, J. Reine. Angew. Math. **314** (1980),

[16] C.D. Popescu, *Base Change for Stark-Type Conjectures over* $\mathbb{Z}$, J. Reine. Angew. Math. **542** (2002), 85-111.

[17] C.D. Popescu, *Rubin's Integral Refinement of the Abelian Stark Conjecture*, In *Stark's conjectures: recent work and directions*, Volume 358 of *Contemp. Math.* , 1-35. Amer. Math. Soc. Providence, RI, 2004.

[18] K. Rubin *A Stark Conjecture "over* $\mathbb{Z}$*" for Abelian L-functions with Multiple Zeros*, K. Rubin, Ann. Inst. Fourier (Grenoble), **46**(1) (1996), 33-62.

[19] *Sage Mathematics Software (Version 2.7)*, W. Stein, The Sage Group, 2007, `http://www.sagemath.org`.

[20] J.W. Sands *Popescu's Conjecture in Multiquadratic Extensions*, In *Stark's conjectures: recent work and directions,* Volume 358 of *Contemp. Math.,* 127-141. Amer. Math. Soc. Providence, RI, 2004.

[21] H.M. Stark *Artin L-functions at* $s = 1$*. II. Artin L-functions with rational characters*, Advances in Math. , **17**(1) (1975), pp. 60-92.

[22] H.M. Stark *Artin L-functions at* $s = 1$*. IV. First Derivatives at* $s = 0$, Advances in Math. , **35**(3) (1980), 197-235.

[23] J. Tate *Les Conjectures de Stark sur les Fonctions L d'Artin en* $s = 0$, Volume 47 of *Progress in Mathematics*, Birkhäuser, Boston, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

[24] E. Weiss *Algebraic Number Theory*, Chelsea Publishing Company, New York, 1963.