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# Monomial Characters of Finite Groups

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# Monomial Characters of Finite Groups

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# Abstract

An abundance of information regarding the structure of a finite group can be obtained by studying its irreducible characters. Of particular interest are monomial characters — those induced from a linear character of some subgroup — since Brauer has shown that any irreducible character of a group can be written as an integral linear combination of monomial characters. Our primary focus is the class of M-groups, those groups all of whose irreducible characters are monomial. A classical theorem of Taketa asserts that an M-group is necessarily solvable, and Dade proved that every solvable group can be embedded as a subgroup of an M-group. After discussing results related to M-groups, we will construct explicit families of solvable groups that cannot be embedded as subnormal subgroups of any M-group. We also discuss groups possessing a unique non-monomial irreducible character, and prove that such a group cannot be simple.

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# Dedication

*For my Family,*
*Mom, Dad, Connor, and Nova.*

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# Chapter 1

# Introduction

Given a representation $X : G \to GL(n, F)$ associated to the action of the finite group $G$ on an $n$-dimensional $F$-vector space via nonsingular linear transformations, the *character* $\chi$ afforded by $X$ is the function defined by

$$\chi(g) = \text{trace}(X(g))$$

for all $g \in G$.[1] When $F = \mathbb{C}$ it turns out that much of the important information about the representation $X$ can be acquired simply by examining the afforded character $\chi$. Furthermore, in this situation one can obtain a wealth of structural information about the group $G$ by considering the set of irreducible characters of $G$. For instance, one can determine relatively quickly whether or not a group is solvable or simple by examining the irreducible characters of the group (recall that every character can be written uniquely as a sum of irreducible characters). Besides these applications to specific groups $G$, characters are useful tools for proving general theorems about finite groups. For instance, we have:

**Theorem 1.0.1.** *(Frobenius) Let $G$ be a finite group and let $H$ be a nontrivial proper subgroup of $G$ such that $H \cap H^g = 1$ for all $g \in G - H$. Then there exists a normal subgroup*

[1] We assume the reader is familiar with the basic elements of group theory and representation theory as covered in [9] parts I and VI.

*$N$ of $G$ such that $NH = G$ and $N \cap H = 1$.*

No known proof of Frobenius' Theorem exists that does not rely on character theory.

In this thesis, we focus on *monomial* characters of a group $G$ — those that are induced from a linear (degree 1) character of a subgroup of $G$. The significance of monomial characters is illustrated in part by Brauer's "characterization of characters," below.

**Theorem 1.0.2.** *(Brauer) Every irreducible character of a group $G$ is a $\mathbb{Z}$-linear combination of monomial characters of $G$.*

Groups all of whose irreducible characters are monomial are referred to as *M-groups.* This family of groups is of interest to character theorists, but has special importance to number theorists as well: a conjecture of Artin, given below, is known to hold for M-groups, but has not been verified for virtually any other class of groups.

Because of its fundamental importance to number theory, and its motivational significance to character theory, we give a precise statement of Artin's conjecture (although it receives no further mention in this thesis). Let $E/F$ be a Galois extension of number fields with Galois group $G$, and let $X : G \to GL(n, \mathbb{C})$ be a representation of $G$ affording the character $\chi$. Define the Artin $L$-series, $L(s, \chi, E/F)$, by

$$L(s, \chi, E/F) = \prod_{\mathcal{P}} \left[ \det\left(1 - N_{F/\mathbb{Q}}(\mathcal{P})^{-s} X|_{V^I}(Frob_{\mathcal{P}})\right) \right]^{-1}, \qquad s \in \mathbb{C},$$

where the product is over all primes $\mathcal{P}$ in $F$, $V^I$ is the subspace of $V$ fixed by the inertia group $I$ of a prime in $E$ over $\mathcal{P}$ and $Frob_{\mathcal{P}}$ is a Frobenius element of $G$ at that prime over $\mathcal{P}$ in $E$. Artin's conjecture states that if the principal character of $G$ is not a constituent of $\chi$ then $L(s, \chi, E/F)$ is an entire function in the complex variable $s$. It can be shown that this conjecture holds whenever $\chi$ is linear. Artin proved that the $L$-series obtained from a character $\varphi$ of a subgroup $H$ of $G$ (for the extension $E/E^H$) is the same as the $L$-series for $\varphi$ induced to $G$. Thus $L(s, \chi, E/F)$ is entire whenever $\chi$ is induced from a linear character of any subgroup, which establishes Artin's conjecture for all monomial characters, hence for any character of an M-group. We refer to [11] for a much more thorough discussion of this important conjecture.

A classical theorem of Taketa asserts that every M-group is solvable. In [2], Berkovich noted that it was not known whether or not a group with a unique nonmonomial irreducible character must also be solvable. Motivated by Berkovich's question, we began the development of this thesis by examining such groups, which we have named *almost M-groups*. We proved via the Classification of Finite Simple Groups that no simple almost M-group exists. It turned out that almost M-groups — if they exist — are necessarily solvable; this is a corollary of a theorem of Qian [21], the proof of which also relies on the Classification. We continued to question the existence of these groups, however. In fact, to date we have yet to find an example of an almost M-group.

Dade has proved that every solvable group can be embedded as a subgroup of an M-group, so by the discussion above, it follows that any almost M-group can be treated as a subgroup of an M-group. Unfortunately, this fact did not provide us with enough traction to show that almost M-groups do not exist. Sometime in the winter of 1973, Huppert asked whether or not every solvable group is a *normal* subgroup of some M-group. Van der Waall showed that the answer to Huppert's question is "no" in general, by proving that $SL(2,3)$ is not a normal subgroup of any M-group [26]. We have expanded on van der Waall's answer by constructing a class of counterexamples to the question: "can every solvable group be embedded as a *subnormal* subgroup of some M-group?"

In Chapter 2 we give a number of preliminary lemmas on groups and characters. We focus especially on monomial characters, M-groups, and results that describe the relation between induction and restriction of characters, such as the theorems of Clifford and Mackey. The original contributions of this thesis are the following theorems, which appear in Chapters 3–5.

**Theorem 4.2.2.** *Let $p$ be an odd prime, let $P$ be an extraspecial $p$-group of order $p^{1+2m}$ and exponent $p$, and let $Q \leq \mathrm{Aut}(P)$ satisfy:*

*(a)* $p \nmid |\,Q\,|$.

*(b) A prime divisor $q$ of $|\,Q\,|$ is a Zsigmondy prime for $(p, 2m)$.*

*(c) $Q$ is a self-normalizing subgroup of $\mathrm{Aut}(P)$, that is, $N_{\mathrm{Aut}(P)}(Q) = Q$.*

*(d) $Q$ acts nontrivially on $Z(P)$.*

*Then $L = P \rtimes Q$ is not a subnormal subgroup of any M-group.*

**Theorem 5.0.6.** *Almost M-groups are not simple.*

We remark that, to the best of our knowledge, the term "almost M-group" is also an original contribution of this thesis. In Chapter 3 we provide examples of small non-M-groups, list basic properties of extraspecial $p$-groups, and give explicit methods for constructing groups that satisfy the hypotheses of Theorem 4.2.2. In Chapter 4, we begin by giving a new, shorter proof of Dade's Embedding Theorem. We then prove Theorem 4.2.2, as well as Theorem 5.0.6 in Chapter 5. Chapter 6 concludes the thesis with a discussion of open problems and possible further directions.

# Chapter 2

# Preliminary Results

We begin by collecting some definitions and facts concerning finite groups and their characters. Although some of the results listed here hold for arbitrary groups and characters, throughout this thesis "group" will mean finite group and characters will always be complex-valued. The notation used is standard, and for the most part is consistent with [9].

In this chapter, when a proof is not given, a reference shall be given instead. Since many of these theorems have been proved in a number of ways, and by a number of different authors, an attempt has been made to reference those proofs that we consider most enlightening. We realize that the reader might prefer a small number of sources instead. If this is the case, we recommend Gorenstein's *Finite Groups* (see [15]) and Isaacs' *Character Theory of Finite Groups* (see [18]).

## 2.1 Group Theory

If $N$ is a normal subgroup of $G$ and $C$ is a characteristic subgroup of $N$, then $C \trianglelefteq G$. However, if $C$ is only *normal* in $N$ then $C$ need not be normal in $G$. In other words, normality is not a transitive relation. We introduce subnormality to remedy this situation.

**Definition 2.1.1.** *A subgroup $H$ of a group $G$ is* subnormal *in $G$ if there are subgroups*

*$H_i$, $0 \le i \le r$, such that $H_0 = H$, $H_r = G$, and $H_i \trianglelefteq H_{i+1}$ for all $i$. If $H$ is subnormal in $G$ we write $H \trianglelefteq\trianglelefteq G$.*

It should be clear that if $H \trianglelefteq\trianglelefteq K$ and $K \trianglelefteq\trianglelefteq G$ then $H \trianglelefteq\trianglelefteq G$, and it is not hard to see that if $H \trianglelefteq\trianglelefteq G$ and $H \le K$ then $H \trianglelefteq\trianglelefteq K$.

In Chapter 3 we will construct a class of groups that are not subnormal subgroups of any M-group. The groups in this class will also satisfy the following definition.

**Definition 2.1.2.** *A group $G$ is* complete *if $Z(G) = 1$ and $\mathrm{Aut}(G) = \mathrm{Inn}(G)$.*

**Lemma 2.1.3.** *Let $N \trianglelefteq G$ and assume that $N$ is complete. Then $G \cong N \times C_G(N)$.*

*Proof.* Set $C = C_G(N)$. Note that $C \trianglelefteq G$ and $N \cap C = Z(N) = 1$, so the subgroup $NC$ of $G$ is isomorphic to the direct product $N \times C$. Let $\varphi : G \to \mathrm{Aut}(N)$ be the homomorphism induced by the action of $G$ on $N$ by conjugation. Then $\varphi(NC) = \mathrm{Aut}(N)$ since every automorphism of $N$ is inner. By the Lattice Isomorphism Theorem we must have $G = NC$, and thus $G \cong N \times C$. $\square$

Given groups $H$ and $K$, the regular wreath product of $H$ by $K$ is denoted $H \wr K$. Basic properties of (general) wreath products are described in [9] — see Exercise 23 in Section 5.5. In particular,

$$H \wr K = \underbrace{(H \times H \times \cdots \times H)}_{|\,K\,|\ \text{factors}} \rtimes K$$

where $K$ acts on the *base group*, $B = H \times H \times \cdots \times H$, by permuting the factors via the regular representation.

In Chapter 4 we shall make use of the so-called *Universal Embedding Theorem for Wreath Products*, stated below.

**Theorem 2.1.4.** *Let $G$ be a group with a normal subgroup $N$ and let $K = G/N$. Then there exists an injective homomorphism $\iota$ from $G$ into the regular wreath product $N \wr K$ such that $\iota$ maps $N$ onto $\iota(G) \cap B$ where $B$ is the base group of $N \wr K$.*

*Proof.* This is established in Section 2.6 of [6]; see Theorem 2.6A. □

Note that this gives a constructive way of embedding $G$ into a larger group in which the extension of $G/N$ by the base group $B$ is split. This is a powerful tool for many purposes.

### 2.1.1 $\pi$-Groups

Let $\pi$ be any set of prime numbers. Then $G$ is a $\pi$-group if every prime divisor of $|\,G\,|$ is an element of $\pi$. For example, the symmetric group $S_4$ is a $\{2,3\}$-group. The complement of $\pi$ in the set of all prime numbers is denoted $\pi'$. Thus $\{2\}'$ is the set of all odd prime numbers. When $\pi = \{p\}$ is singleton we shall simply write $p$ for $\pi$ and $p'$ for $\pi'$.

Recall that if $G$ is a group and $\pi$ is a set of prime numbers then a subgroup $H$ of $G$ is a *Hall $\pi$-subgroup* (or simply a *Hall subgroup*) if $H$ is a $\pi$-group and $(|\,H\,|, |\,G : H\,|) = 1$, that is, the order of $H$ and its index in $G$ are relatively prime. If $\pi = \{p\}$ then a Hall $\pi$-subgroup of $G$ is the same as a Sylow $p$-subgroup of $G$. A standard result due to Philip Hall states that if $G$ is solvable then for any set $\pi$ of primes, $G$ has a Hall $\pi$-subgroup and any two Hall $\pi$-subgroups are conjugate in $G$ (see Exercise 33 in Section 6.1 of [9]).

The well-known lemma below is often referred to as *Frattini's Argument.*

**Lemma 2.1.5.** *(Frattini) Let $N \trianglelefteq G$ and let $H$ be a Hall $\pi$-subgroup of $N$. If all Hall $\pi$-subgroups of $N$ are conjugate in $N$, then $G = N_G(H)N$.*

If $N$ and $M$ are two normal $\pi$-subgroups of a group $G$, then their product $NM$ is also a normal $\pi$-subgroup. Thus $G$ has a unique largest normal $\pi$-subgroup, and it is denoted $O_\pi(G)$. Note that $O_\pi(G)$ is in fact characteristic in $G$. If $N \trianglelefteq G$ then $O_\pi(N) = O_\pi(G) \cap N$. Repeated applications of this fact produce the same equality if $N$ is only subnormal in $G$.

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$, is the intersection of all maximal subgroups of $G$. When $G$ is a $p$-group, the Frattini subgroup plays a role similar to that of the commutator subgroup; namely, if $N \trianglelefteq G$ then $G/N$ is elementary abelian if and only if

$\Phi(G) \leq N$.

**Theorem 2.1.6.** *(Burnside's Basis Theorem) Let $p$ be a prime, let $P$ be a $p$-group, and let $\overline{P} = P/\Phi(P)$.*

*(a) A set $x_1, x_2, \ldots, x_n$ is a minimal generating set for $P$ if and only if $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}$ is a basis for $\overline{P}$ considered as a vector space over $\mathbb{F}_p$.*

*(b) Let $\alpha \in \mathrm{Aut}(P)$ have order relatively prime to $p$ and suppose that $\alpha$ acts trivially on $\overline{P}$. Then $\alpha$ is the identity automorphism of $P$.*

*Proof.* See Theorem 1.4 in Chapter 5 of [15]. □

In Chapters 3 and 4 we shall make use of the following lemma.

**Lemma 2.1.7.** *If $A$ is a $p'$-group of automorphisms of the $p$-group $P$, then*

$$P = C_P(A) \cdot [P, A],$$

*where $C_P(A)$ is the subgroup of $P$ consisting of those elements fixed by every automorphism in $A$ and $[P, A]$ is the subgroup of $P$ generated by all elements of the form $x^{-1}\alpha(x)$, $x \in P$ and $\alpha \in A$. We also have*

$$[P, A] = [P, A, A],$$

*where $[P, A, A] = [[P, A], A]$ by definition.*

*Proof.* See Theorems 3.5 and 3.6 of Chapter 5 in [15]. □

## 2.2 Character Theory

If $G$ is a group, we shall write $\mathrm{Irr}(G)$ for the set of (complex) irreducible characters of $G$. If $N \trianglelefteq G$, then the set $\{\chi \in \mathrm{Irr}(G) : N \leq \ker \chi\}$ can be identified with the irreducible characters of the quotient group $G/N$, and thus we denote this set by $\mathrm{Irr}(G/N)$. The principal character of $G$ is represented by $1_G$ and $(\cdot,\cdot)$ is the usual Hermitian inner product on the space of $\mathbb{C}$-valued class functions of $G$.

If $H \leq G$ and $\chi$ is any character of $G$, we write $\chi_H$ for the restriction of $\chi$ to $H$. If $\varphi$ is a character of $H$ then we write $\varphi^G$ for the character of $G$ induced from $\varphi$. The values of the induced character are given by the formula

$$\varphi^G(g) = \frac{1}{|\,H\,|} \sum_{\substack{x \in G \\ g^x \in H}} \varphi(g^x),$$

where $g$ is any element of $G$ and $g^x = x^{-1}gx$. The lemma below follows immediately from this formula.

**Lemma 2.2.1.** *Let $H \leq G$, let $\varphi$ be a character of $H$, and let $\chi$ be a character of $G$. Then $(\varphi\chi_H)^G = \varphi^G\chi$.*

If $H$ and $K$ are groups then

$$\mathrm{Irr}(H \times K) = \{\varphi \otimes \psi : \varphi \in \mathrm{Irr}(H) \text{ and } \psi \in \mathrm{Irr}(K)\},$$

and if $\varphi \otimes \psi$ is an irreducible character of $H \times K$ then $(\varphi \otimes \psi)(h,k) = \varphi(h)\psi(k)$ for all $h \in H$ and $k \in K$ (see Exercise 27 in Section 18.3 of [9]).

Note that if both $\varphi$ and $\psi$ are *faithful* irreducible characters then by Schur's Lemma $\varphi \otimes \psi$ is a faithful character of $H \times K$ if and only if $(|Z(H)|, |Z(K)|) = 1$.

### 2.2.1 Clifford's Theorem and Consequences

Given a group $G$ with subgroup $H$ and character $\chi$, one often wishes to express the restriction $\chi_H$ in terms of its irreducible constituents. Unfortunately, this can be quite difficult even for specific $G$ and $H$. This subsection shows that the situation is much different when

$H$ is normal in $G$.

Let $\theta$ be a character of the normal subgroup $N$ of $G$. If $g \in G$, define $\theta^g$ by $\theta^g(n) = \theta(gng^{-1})$ for all $n \in N$. Then $\theta^g$ is a character of $N$, called the *conjugate of* $\theta$ *by* $g$, which is irreducible if and only if $\theta$ is. The map $\theta \mapsto \theta^g$ is a right group action of $G$ on the set of characters of $N$. Irreducible characters $\theta_1$ and $\theta_2$ of $N$ are conjugate if and only if $\theta_1^G = \theta_2^G$. An *invariant* character of $N$ is one that is fixed under conjugation by $G$.

**Theorem 2.2.2.** *(Clifford) Let $N$ be a normal subgroup of $G$, let $\chi \in \mathrm{Irr}(G)$, and let $\theta$ be an irreducible constituent of $\chi_N$. Denote the distinct $G$-conjugate characters of $\theta$ by $\theta = \theta_1, \theta_2, \ldots, \theta_t$. Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i,$$

*where $e = (\chi_N, \theta)$.*

*Proof.* We have already noted that two irreducible characters of $N$ are $G$-conjugate if and only if they induce the same character of $G$. Thus by Frobenius Reciprocity $(\chi_N, \theta) = (\chi_N, \theta_i)$ for all $i$, and it remains to show that $\theta_1, \theta_2, \ldots, \theta_t$ are all of the irreducible constituents of $\chi_N$. This follows from the formula for the values of an induced character. In fact, for all $n \in N$ we have

$$\theta^G(n) = \frac{1}{|\,N\,|} \sum_{\substack{g \in G \\ n^g \in N}} \theta(n^g) = \frac{1}{|\,N\,|} \sum_{g \in G} \theta^g(n),$$

so the irreducible constituents of $(\theta^G)_N$ are exactly the conjugates of $\theta$. Finally, if $\psi \in \mathrm{Irr}(N)$ is not conjugate to $\theta$ then

$$(\chi_N, \psi) = (\chi, \psi^G) \leq (\theta^G, \psi^G) = ((\theta^G)_N, \psi) = 0.$$

This completes the proof. $\square$

When $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, the subgroup

$$I_G(\theta) = \{g \in G : \theta^g = \theta\}$$

is called the *inertia group of* $\theta$ *in* $G$. Note that we always have $N \leq I_G(\theta)$. Assuming the hypotheses of Theorem 2.2.2, we also have $t = |G : I_G(\theta)|$ by the Orbit-Stabilizer Theorem. In particular, $t$ must divide the index of $N$ in $G$.

The second part of Clifford's Theorem is the following.

**Theorem 2.2.3.** *Let $\theta$ be an irreducible character of the normal subgroup $N$ of $G$ and let $T = I_G(\theta)$. Then for each irreducible constituent $\chi$ of $\theta^G$, there is a unique character $\zeta \in \mathrm{Irr}(T)$ such that $(\zeta, \theta^T)$ and $(\zeta^G, \chi)$ are both nonzero. For this $\zeta$ we have $\zeta^G = \chi$ and $\zeta_N = e\theta$, where $e = (\chi_N, \theta)$.*

*Proof.* This is Theorem 6.11 of [18]. □

We shall make frequent use of the following corollary.

**Corollary 2.2.4.** *Let $\chi$ be an irreducible character of $G$. Let $N \trianglelefteq G$ and let $\theta$ be an irreducible constituent of $\chi_N$. Write*

$$\chi_N = e \sum_{i=1}^{t} \theta_i,$$

*where $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct $G$-conjugates of $\theta$. If $I_G(\theta)/N$ is cyclic, then $e = 1$. In particular, if $N$ has prime index $p$ in $G$ then either $\chi_N$ is irreducible or $\chi_N = \sum_{i=1}^{p} \theta_i$, with $\theta_i^G = \chi$ for each $i$. Furthermore, if $N \trianglelefteq G$ with $G/N$ cyclic, then each invariant character of $N$ extends to $G$.*

*Proof.* See Theorem 9.12 of [10]. □

A related result, first shown by Gallagher in [14], is given below.

**Theorem 2.2.5.** *(Gallagher) If $N$ is a normal Hall subgroup of $G$, then each invariant irreducible character of $N$ extends to a character of $G$.*

If $N \trianglelefteq G$, we define the *regular character of* $G/N$, denoted $\rho_{G/N}$, by

$$\rho_{G/N} = \sum_{\psi \in \mathrm{Irr}(G/N)} \psi(1)\psi.$$

Note that $\rho_{G/N}$ as defined is a character of $G$. However, $\rho_{G/N}$ may be identified with the regular character of the quotient group $G/N$ in the same way we identify the irreducible characters of this quotient with the set $\mathrm{Irr}(G/N)$. This justifies our choice of notation.

**Theorem 2.2.6.** *(Gallagher) Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct $\beta$, and are all of the irreducible constituents of $\theta^G$. In fact, $\theta^G = \rho_{G/N}\chi$.*

*Proof.* See Theorem 6.17 of [18]. □

### 2.2.2 Monomial Characters and M-Groups

**Definition 2.2.7.** *A character $\chi$ of $G$ is* monomial *if it is induced from a linear character of some subgroup of $G$. If all irreducible characters of $G$ are monomial, then $G$ is an* M-group*.*

We give an equivalent module-theoretic description of monomial characters.

**Lemma 2.2.8.** *Let $V$ be a $\mathbb{C}G$-module affording the character $\chi$. Then the following are equivalent:*

*(a) $\chi$ is monomial.*

*(b) $V$ is a direct sum of one-dimensional subspaces that are permuted transitively by $G$.*

Consequently, if the character $\chi$ afforded by the representation $X$ of $G$ is monomial, then for all $g \in G$ the matrix $X(g)$ has exactly one nonzero entry in each row and column. The converse does not hold in general.

Perhaps the most well-known fact about M-groups is that they are necessarily solvable. This was first shown by Taketa [25], but the proof given here is taken from Feit [10].

**Theorem 2.2.9.** *(Taketa) Every M-group is solvable.*

*Proof.* Let $G$ be an M-group and suppose that $G$ is not solvable. Denote the terminal member of the derived series of $G$ by $G^\infty$. Then

$$G^\infty \not\leq \bigcap_{\chi \in \mathrm{Irr}(G)} \ker \chi = 1,$$

so we may choose an irreducible character $\chi$ of $G$ of minimum degree that does not contain $G^\infty$ in its kernel. Note that $\chi$ must be nonlinear. Since $\chi$ is monomial, there exists a subgroup $H$ of $G$ and a linear character $\lambda$ of $H$ such that $\lambda^G = \chi$. Now, $(1_H)^G = 1_G + \psi$ for some (possibly reducible) character $\psi$ of $G$. We have $\psi(1) < |G : H| = \chi(1)$, so by minimality of $\chi$ every irreducible constituent of $\psi$ contains $G^\infty$ in its kernel. It follows that $G^\infty \leq \ker(1_H)^G \leq H$. In fact, $G^\infty = (G^\infty)' \leq H'$, so $G^\infty$ is contained in the kernel of $\lambda$. But this forces $G^\infty \leq \ker \chi$, a contradiction. □

It is not the case that every solvable group is an M-group, the smallest example being the group $SL(2,3)$ which has order 24. Indeed, $SL(2,3)$ has 3 irreducible characters of degree 2, but no subgroup of index 2 (see Section 3.1). However, if a group is supersolvable then it is necessarily an M-group. Recall that a group $G$ is said to be *supersolvable* if there exists a series

$$1 = N_0 \leq N_1 \leq \cdots \leq N_r = G$$

such that each $N_i$ is normal in the whole group $G$ and each factor $N_i/N_{i-1}$ is cyclic (i.e., $G$ has a *chief series* with cyclic factors).

**Theorem 2.2.10.** *Supersolvable groups are M-groups.*

*Proof.* This is a corollary of Theorem 6.22 in [18]. □

**Lemma 2.2.11.** *Let $N \trianglelefteq G$. Then $G/N$ is an M-group if and only if each character in* $\mathrm{Irr}(G/N)$ *is monomial.*

*Proof.* Let $V$ be the $\mathbb{C}G$-module affording $\chi \in \mathrm{Irr}(G/N)$. Then by Lemma 2.2.8, $\chi$ is monomial if and only if $V$ is the direct sum of one-dimensional subspaces that are permuted transitively by $G$, hence by $G/N$ since $N \leq \ker \chi$. Equivalently, $V$ is an irreducible $\mathbb{C}(G/N)$-module that affords a monomial character. The lemma follows. □

**Lemma 2.2.12.** *Groups $H$ and $K$ are M-groups if and only if the direct product $H \times K$ is an M-group.*

*Proof.* This is a consequence of Lemmas 2.2.8 and 2.2.11. □

In contrast to the class of solvable groups, subgroups of M-groups need not be M-groups. In fact, even normal subgroups of M-groups are not necessarily M-groups, as Dade has shown [5]. However, if a normal subgroup of an M-group is Hall then it must be an M-group. This was first shown by Dornhoff [7].

**Theorem 2.2.13.** *(Dornhoff) Normal Hall subgroups of M-groups are M-groups.*

We mention that Fukushima has provided an example of an M-group containing (non-normal) Hall subgroups that are not themselves M-groups [13].

### 2.2.3 Mackey's Theorem

Mackey's Theorem describes what happens when we induce a character from one subgroup and then restrict it to another. Thankfully, the subgroups in question need not be normal.

Let $K$ and $H$ be subgroups of the group $G$. For each $g \in G$, define the $(K, H)$-*double coset of $g$ in $G$* to be the set

$$KgH = \{kgh : k \in K \text{ and } h \in H\}.$$

The basic properties of $(K, H)$-double cosets are listed in Exercise 10, Section 4.1, of [9].

If $\varphi$ is a character of the subgroup $H$ of $G$ and $g \in G$, then ${}^g\varphi$, defined by

$$ {}^g\varphi(ghg^{-1}) = \varphi(h), $$

is a character of the conjugate $gHg^{-1}$.

**Theorem 2.2.14.** *(Mackey) Let $H$ and $K$ be subgroups of $G$ and let $T$ be a set of $(K, H)$-double coset representatives. For each $t \in T$, set $H_t = tHt^{-1} \cap K$. If $\varphi$ is a character of $H$, then*

$$ (\varphi^G)_K = \sum_{t \in T} ({}^t\varphi_{H_t})^K. $$

*Proof.* A proof is given in Section 7.3 of [23]. $\square$

**Corollary 2.2.15.** *If $H$ and $K$ are subgroups of $G$ with $G = HK$ and if $\varphi$ is a character of $H$ then $(\varphi^G)_K = (\varphi_{H \cap K})^K$.*

*Proof.* In the notation of Theorem 2.2.14, take $T = \{1\}$. $\square$

**Corollary 2.2.16.** *Let $\chi$ be a monomial character of $G$ and let $K \leq G$. If $\chi_K$ is irreducible then it is also monomial.*

*Proof.* There exists a subgroup $H \leq G$ and a linear character $\lambda$ of $H$ such that $\lambda^G = \chi$. We have $(\lambda^G)_K \in \mathrm{Irr}(K)$, so by Mackey's Theorem $G = HK$. By Corollary 2.2.15,

$$ \chi_K = (\lambda^G)_K = (\lambda_{H \cap K})^K. $$

We see that $\chi_K$ is monomial. $\square$

### 2.2.4 Tensor Induction

Following Isaacs [17], in this section we define *tensor-induction* of characters. As the name suggests, tensor-induction is a way of constructing a character of a group $G$ from a character $\varphi$ of a subgroup $H$. The character of $G$ obtained through tensor-induction shall be denoted $\varphi^{\otimes G}$.

Let $H \leq G$ and fix a set $T$ of representatives for the right cosets of $H$ in $G$. Define a right action of $G$ on $T$ by $t \cdot g = t' \in T$ if and only if $Htg = Ht'$. For a fixed $g \in G$, let $T_0$ be a set of representatives for the orbits in $T$ under the action of $\langle g \rangle$, and for $t \in T$, let $n(t)$ denote the size of the $\langle g \rangle$-orbit containing $t$. If $\varphi$ is a character of $H$, define $\varphi^{\otimes G}$ by

$$\varphi^{\otimes G}(g) = \prod_{t \in T_0} \varphi(tg^{n(t)}t^{-1})$$

for all $g \in G$.

It is, perhaps, not obvious that $\varphi^{\otimes G}$ as defined is a character of $G$. This fact, as well as the lemma below, are established in [17]; we omit the proofs for brevity.

**Lemma 2.2.17.** *Let $N \leq H \leq G$ with $N \trianglelefteq G$ and let $\varphi$ be a character of $H$. Then for $n \in N$, we have*

$$\varphi^{\otimes G}(n) = \prod_{t \in T} \varphi(tnt^{-1})$$

*where $T$ is a set of representatives for the right cosets of $H$ in $G$.*

The reason we have introduced tensor-induction of characters is so that we may apply the following theorem, which is a slightly modified version of a result due to Bianchi, Chillag, Lewis, and Pacifici [1].

**Theorem 2.2.18.** *Let $L = L_1$ be a complete group and let $N = L_1 \times L_2 \times \cdots \times L_s$ be a normal subgroup of $G$ such that $L_1, L_2, \ldots, L_s$ are all of the conjugates of $L_1$ (so in particular, $G$ permutes the subgroups $L_i \cong L$ transitively by conjugation). Let $\sigma \in \mathrm{Irr}(L)$.*

*Then* $\sigma \otimes \sigma \otimes \cdots \otimes \sigma \in \mathrm{Irr}(N)$ *extends to an irreducible character of* $G$*.*

*Proof.* Let $H = N_G(L_1)$ and $C = C_G(L_1)$. By the Orbit-Stabilizer Theorem, $|G : H| = s$. Let $T$ be a set of representatives for the right cosets of $H$ in $G$. We may write $T = \{x_1, x_2, \ldots, x_s\}$ where $L_1^{x_i} = L_i$ for all $i$. Let $\sigma_i \in \mathrm{Irr}(N)$ be the irreducible character of $N$ whose $i^{\text{th}}$ component is $\sigma$ and whose $j^{\text{th}}$ component is $1_{L_j}$ if $j \neq i$.

Since $C_H(L_1) = C$, we have $H = L_1 \times C$ by Lemma 2.1.3. Let $\theta = \sigma \otimes 1_C \in \mathrm{Irr}(H)$ and note that $\theta_N = \sigma_1$. Write $\chi = \theta^{\otimes G}$. Then by Lemma 2.2.17, for any $y \in N$ we have

$$\chi(y) = \theta^{\otimes G}(y) = \prod_{i=1}^{s} \theta(x_i y x_i^{-1}) = \prod_{i=1}^{s} \sigma_1(x_i y x_i^{-1}) = \prod_{i=1}^{s} \sigma_i(y) = (\sigma \otimes \sigma \otimes \cdots \otimes \sigma)(y).$$

So $\chi_N = \sigma \otimes \sigma \otimes \cdots \otimes \sigma$, which proves the theorem. □

Under the notation of Theorem 2.2.18, if $Z(L) = 1$ then by the Krull-Schmidt Theorem (see [16], Chapter 12) $G$ must permute the subgroups $L_1, L_2, \ldots, L_2$ by conjugation, i.e, each $G$-conjugate of $L_1$ is among the $L_i$. This will be the situation we encounter in this thesis.

## 2.3 Number Theory

**Definition 2.3.1.** *If* $a$ *and* $n$ *are integers greater than* $1$*, then a prime* $q$ *is a* Zsigmondy prime for $(a, n)$ *if* $q \mid a^n - 1$ *but* $q \nmid a^k - 1$ *for all integers* $k$ *in the range* $0 < k < n$*.*

**Theorem 2.3.2.** *(Bang-Zsigmondy) Let* $a$ *and* $n$ *be integers greater than* $1$*. Then a Zsigmondy prime* $q$ *for* $(a, n)$ *exists except in the following cases:*

*(a)* $n = 2$ *and* $a = 2^r - 1$*, where* $r \geq 2$*.*

*(b)* $n = 6$ *and* $a = 2$*.*

*Proof.* A proof is given in [22]. □

# Chapter 3

# Examples and Constructions

## 3.1 The Characters of $SL(2,3)$ and $GL(2,3)$

We have already mentioned that $SL(2,3)$ is the smallest solvable non-M-group. Another example of such a group is $GL(2,3)$, which contains $SL(2,3)$ as a normal subgroup of index 2. In this section we give the character tables and describe the nonmonomial irreducible characters of both groups. These computations are straightforward so the (lengthy) details are omitted.

In the character tables below, each conjugacy class is represented by the order of its elements. Subscripts are used when there are two or more conjugacy classes whose elements have the same order.

As abstract groups it is easy to see that

$$SL(2,3) \cong Q_8 \rtimes Z_3 \qquad \text{and} \qquad GL(2,3) \cong Q_8 \rtimes S_3$$

where the faithful semidirect product actions are uniquely determined. For ease, let $S = SL(2,3)$ and $G = GL(2,3)$. Note that a Sylow 2-subgroup of $S$ is normal and isomorphic to $Q_8$, while a Sylow 2-subgroup of $G$ is non-normal and isomorphic to the quasidihedral group

| Class: | 1 | 2 | $3_1$ | $3_2$ | 4 | $6_1$ | $6_2$ |
|---|---|---|---|---|---|---|---|
| Size: | 1 | 1 | 4 | 4 | 6 | 4 | 4 |
| $\theta_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\theta_2$ | 1 | 1 | $\omega$ | $\overline{\omega}$ | 1 | $\omega$ | $\overline{\omega}$ |
| $\theta_3$ | 1 | 1 | $\overline{\omega}$ | $\omega$ | 1 | $\overline{\omega}$ | $\omega$ |
| $\theta_4$ | 2 | $-2$ | $-1$ | $-1$ | 0 | 1 | 1 |
| $\theta_5$ | 2 | $-2$ | $-\omega$ | $-\overline{\omega}$ | 0 | $\omega$ | $\overline{\omega}$ |
| $\theta_6$ | 2 | $-2$ | $-\overline{\omega}$ | $-\omega$ | 0 | $\overline{\omega}$ | $\omega$ |
| $\theta_7$ | 3 | 3 | 0 | 0 | $-1$ | 0 | 0 |

*Table 3.1: The character table of $SL(2,3)$. Here $\omega = e^{2\pi i/3}$.*

$QD_{16}$. The centers of both $S$ and $G$ are generated by the negative identity matrix, hence $Z(S) = Z(G) \cong Z_2$. The projective groups $PSL(2,3) = S/Z(S)$ and $PGL(2,3) = G/Z(G)$ act naturally on the four points on the projective line over $\mathbb{F}_3$ giving:

$$PSL(2,3) \cong A_4 \qquad \text{and} \qquad PGL(2,3) \cong S_4.$$

So the nonfaithful irreducible characters of $S$ and $G$ are easily computed from the character tables for $A_4$ and $S_4$ (which are given in Chapter 19 of [9], for example).

By studying Table 3.1 one observes that $S$ has no (normal) subgroup of index 2; for otherwise, $S$ would have a nonprincipal linear character with values in $\{\pm 1\}$. Hence each degree 2 irreducible character of $S$ is nonmonomial. In fact, these are all of the nonmonomial characters in $\mathrm{Irr}(S)$. To see that the unique degree 3 character $\theta_7$ of $S$ is monomial, note that a Sylow 2-subgroup, say $Q$, of $S$ is normal and has index 3. If $\chi \in \mathrm{Irr}(S)$ with $\chi(1) = 3$, then $\chi_Q$ cannot be irreducible since 3 does not divide $|\,Q\,|$, so by Corollary 2.2.4, the restriction $\chi_Q$ is the sum of three distinct characters of $Q$. Each constituent is necessarily linear, and each induces $\chi$. (Alternatively, from the character table one sees that $\ker\theta_7 = Z(S)$ so that $\theta_7 \in \mathrm{Irr}(S/Z(S))$. Since $S/Z(S) \cong A_4$ is an M-group, $\theta_7$ is monomial.)

We briefly describe how the character table of $G$ can be obtained from that of $S$. Since $S$ is a normal subgroup of $G$ of index 2, by Corollary 2.2.4 each invariant character of $S$

extends to $G$. Furthermore, Theorem 2.2.6 states that each invariant character extends in exactly 2 ways. Now, $S$ has 3 rational-valued irreducible characters $\theta_1$, $\theta_4$, and $\theta_7$, and each is the unique rational character of its degree. It is necessary, then, that each be invariant in $G$. In fact, these are the only $G$-invariant irreducible characters of $S$. The 4 remaining characters of $S$ form 2 orbits under the action of $G$, so by inducing we obtain 2 distinct irreducible characters of $G$. This accounts for all 8 irreducible characters of $G$.

| Class: | 1 | $2_1$ | $2_2$ | 3 | 4 | 6 | $8_1$ | $8_2$ |
|---|---|---|---|---|---|---|---|---|
| Size: | 1 | 1 | 12 | 8 | 6 | 8 | 6 | 6 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 2 | 2 | 0 | $-1$ | 2 | $-1$ | 0 | 0 |
| $\chi_4$ | 2 | $-2$ | 0 | $-1$ | 0 | 1 | $\alpha$ | $-\alpha$ |
| $\chi_5$ | 2 | $-2$ | 0 | $-1$ | 0 | 1 | $-\alpha$ | $\alpha$ |
| $\chi_6$ | 3 | 3 | 1 | 0 | $-1$ | 0 | $-1$ | $-1$ |
| $\chi_7$ | 3 | 3 | $-1$ | 0 | $-1$ | 0 | 1 | 1 |
| $\chi_8$ | 4 | $-4$ | 0 | 1 | 0 | $-1$ | 0 | 0 |

*Table 3.2: The character table of $GL(2,3)$. Here $\alpha = \sqrt{-2}$.*

It is obvious that the linear characters $\chi_1$ and $\chi_2$ of $G$ are monomial, and we have seen that $\chi_3$ is induced from the linear characters $\theta_2$ and $\theta_3$ of $S$, i.e., that $\chi_3$ is also monomial. Since $S$ is the unique subgroup of $G$ having index 2 and both $\chi_4$ and $\chi_5$ are extensions of $\theta_4$, neither can be monomial. That $\chi_4$ and $\chi_5$ are the only nonmonomial characters in $\mathrm{Irr}(G)$ is easily seen. We have $\chi_6, \chi_7 \in \mathrm{Irr}(G/Z(G))$ monomial because $G/Z(G) \cong S_4$ is an M-group, and $\chi_8$ is induced from a linear character $\lambda$ of a subgroup that is isomorphic to $Z(G) \times S_3$, where $Z(G) \not\leq \ker \lambda$.

There is another group, $G_1$, containing $SL(2,3)$ as a subgroup of index 2, but with Sylow 2-subgroups of type $Q_{16}$. Both $GL(2,3)$ and $G_1$ are the 2-fold ("universal") covering groups of the symmetric group $S_4$ (discovered by Schur). The group $G_1$ also has exactly two nonmonomial characters, and the numbers and degrees of its irreducible characters is the same as for $GL(2,3)$ (although $G_1$ has one class of involutions and two classes of elements of order 4).

By the observations above we see that the number of nonmonomial irreducible characters of $G$ is less than that of $S$. One might wonder if $G$ is a normal subgroup of a group with even fewer nonmonomial characters; for instance, an M-group. We shall return to this question later.

## 3.2 Extraspecial $p$-Groups

Extraspecial $p$-groups play a fundamental role in our constructions.[1] In this section we examine the structure and automorphisms of these groups, as well as their irreducible characters.

**Definition 3.2.1.** *A $p$-group $P$ is* extraspecial *if $Z(P) = P' \cong Z_p$.*

For example, the dihedral and quaternion groups of order 8 are both extraspecial. In fact, any nonabelian group of order $p^3$ is extraspecial.

The following lemma collects some elementary facts about extraspecial $p$-groups.

**Lemma 3.2.2.** *Let $P$ be an extraspecial $p$-group.*

*(a)* $|\,P\,| = p^{1+2m}$ *for some* $m \in \mathbb{N}$.

*(b) The quotient $P/Z(P)$ is elementary abelian. In particular, $P/Z(P) \cong \mathrm{Inn}(P)$ is a vector space of dimension $2m$ over $\mathbb{F}_p$.*

*(c) $Z(P)$ is the unique minimal normal subgroup of $P$.*

*(d) The map $(x, y) \mapsto [x, y]$ is a nondegenerate symplectic form on $P/Z(P)$.*

We note that part (b) of Lemma 3.2.2 is equivalent to stating that $Z(P) = \Phi(P)$, the Frattini subgroup of $P$.

Theorem 3.2.3 below describes the automorphism group of an extraspecial $p$-group.

[1] Throughout this section, $p$ denotes a prime.

**Theorem 3.2.3.** *Let $P$ be an extraspecial $p$-group of order $p^{1+2m}$. Set $A = \mathrm{Aut}(P)$ and*

$$C = C_A(Z(P)) = \{\alpha \in A : \alpha \text{ is the identity on } Z(P)\}.$$

*Then $A \cong C \rtimes Z_{p-1}$, and $C/\mathrm{Inn}(P)$ is isomorphic to a subgroup of the symplectic group $Sp(2m,p)$. Furthermore, if $p$ is odd and $P$ has exponent $p$, then $C/\mathrm{Inn}(P) \cong Sp(2m,p)$ and $C_A(P/Z(P)) = \mathrm{Inn}(P)$.*

*Proof.* We refer to [28] for a proof. □

The only information about the symplectic group $Sp(2m,p)$ required for our constructions is the order formula:

$$|\, Sp(2m,p)\,| = p^{m^2}\prod_{i=1}^{m}(p^{2i}-1).$$

Let $P$ denote an extraspecial $p$-group of order $p^{1+2m}$ once again and let $Z = Z(P)$. Since $Z$ is the unique minimal normal subgroup of $P$, any irreducible character of $P$ is either faithful or contains $Z$ in its kernel. There are precisely $|\,P : Z\,| = p^{2m}$ characters of the second type, since $P/Z$ is abelian; in particular, any such character is linear. It is not hard to see that $P$ has $p-1$ faithful irreducible characters, each of degree $p^m$. Let $\varphi \in \mathrm{Irr}(P)$ be faithful. Then by Schur's Lemma $\varphi_Z = p^m\lambda$ for some (nonprincipal) $\lambda \in \mathrm{Irr}(Z)$. Hence $(\varphi_Z, \varphi_Z) = p^{2m} = |P : Z|$, so by calculating $(\varphi, \varphi)$ one sees that $\varphi$ must vanish on $P - Z$. If we let $z \in P$ be a generator for $Z$, it follows immediately that each faithful irreducible character of $P$ is uniquely determined by its value on $z$. We summarize these facts in the theorem below.

**Theorem 3.2.4.** *Let $P$ be an extraspecial $p$-group of order $p^{1+2m}$. Then $P$ has $p^{2m}$ linear characters and $p-1$ faithful irreducible characters. Each faithful irreducible character has degree $p^m$, vanishes off of $Z(P)$, and is uniquely determined by its value on a (fixed)* *generator for $Z(P)$. Furthermore, all faithful irreducible characters are Galois conjugate.*

We also observe that any extraspecial $p$-group is supersolvable, hence is an M-group by Theorem 2.2.10.

## 3.3 A Complete Non-M-Group

The non-M-group $SL(2,3)$ is isomorphic to the semidirect product of $Q_8$ and $Z_3$, hence is a split extension of a $p'$-group by an extraspecial $p$-group. In this section and the next, we construct more groups of this form and show that they are also non-M-groups.

Let $P$ be the extraspecial group of order $27 = 3^3$ and exponent 3. Set $Z = Z(P)$ and $A = \mathrm{Aut}(P)$. By Theorem 3.2.3, we have $A \cong C_A(Z) \rtimes Z_2$ and $|\,A\,| = 2^4 \cdot 3^3$. Let $Q \in \mathrm{Syl}_2(A)$. Note that $Q$ does not centralize $Z$ since $|\,A : C_A(Z)\,| = 2$. Note also that $Q$ has an element of order 8 since $\overline{P} = P/Z$ has an automorphism of order 8 (in fact, $Q \cong QD_{16}$).

Form the semidirect product $L = P \rtimes Q$. We show that $L$ has nonmonomial irreducible characters. By Theorem 3.2.4, $P$ has two faithful irreducible characters of degree 3 that are uniquely determined by their values on a generator for $Z$. Let $\theta$ be such a character and set $T = I_L(\theta)$. Since $Q$ does not centralize $Z$, $|\,L : T\,| = 2$ by the Orbit-Stabilizer Theorem. Now $P$ is a normal Hall subgroup of $T$, so by Theorem 2.2.5 there exists $\zeta \in \mathrm{Irr}(T)$ such that $\zeta_P = \theta$. We have $\zeta^G = \chi \in \mathrm{Irr}(L)$ by Theorem 2.2.3. The character $\chi$ cannot be monomial: if it were, there would exist a subgroup $H \leq L$ of index $\zeta^G(1) = 6$. Then we would have $H \cap P \trianglelefteq H$ and $|\,P : H \cap P\,| = 3$, so by Maschke $H$ would act reducibly on $\overline{P}$. The action of $Q$ is also faithful by Theorem 2.1.6, so since $Q$ has an element of order 8, there must be an element of order 4 in $H$ that acts faithfully on $\overline{P}$ as a $2 \times 2$ diagonal matrix over $\mathbb{F}_3$. This is impossible. Thus $\chi$ is not a monomial character and $L$ is not an M-group.

Unlike $SL(2,3)$, the group $L$ is complete: it is not hard to see that $Z$ is the unique minimal normal subgroup of $L$, and since $Q$ does not centralize $Z$ it follows that $Z(L) = 1$. Thus $L$ embeds as a normal subgroup of $\mathrm{Aut}(L)$, and since $P$ is characteristic in $L$ by Sylow, we have $P \trianglelefteq \mathrm{Aut}(L)$ also. Let $\varphi : \mathrm{Aut}(L) \to A$ be the homomorphism afforded by the action of $\mathrm{Aut}(L)$ on $P$ by restriction of an automorphism of $L$ to $P$. Then the kernel of $\varphi$ is $Z$. If $\varphi$ were surjective, we would have $L/Z \trianglelefteq A$, hence a Sylow 2-subgroup of $\mathrm{Out}(P) \cong GL(2,3)$ would be normal, a contradiction. Thus $\varphi$ is not surjective, and so $L = \mathrm{Aut}(L)$.

## 3.4 A Class of Complete Non-M-Groups

In this section we extend (and modify) the ideas of the previous construction.

Let $p$ be an odd prime, let $m \geq 2$, and let $P$ denote the extraspecial $p$-group of order $p^{1+2m}$ and exponent $p$. Set $A = \mathrm{Aut}(P)$ and $Z = Z(P)$. By Theorem 3.2.3,

$$A \cong C_A(Z) \rtimes Z_{p-1} \qquad \text{and} \qquad |\, A \,| = p^{m(m+2)}(p-1)\prod_{i=1}^{m}(p^{2i}-1).$$

Let $q$ be a Zsigmondy prime for $(p, 2m)$ — which exists by Theorem 2.3.2 — and assume that $p \nmid q-1$ (as an example, take $p = 3$, $m = 2$, and $q = 5$). Then $A$ has nontrivial Sylow $q$-subgroups. Let $Q_1 \in \mathrm{Syl}_q(A)$. Since $q \nmid p-1$, $Q_1$ is also a Sylow $q$-subgroup of $C_A(Z)$. Set $Q = N_A(Q_1)$. Note that because $Q_1$ is characteristic in $Q$,

$$Q \textit{ is a self-normalizing subgroup of } A = \mathrm{Aut}(P).$$

Since $Q_1 \leq C_A(Z) \trianglelefteq A$, by Frattini's Argument $A = C_A(Z)Q$. It follows that

$$Q \textit{ acts nontrivially on } Z = Z(P).$$

**Lemma 3.4.1.** *If $x$ is an automorphism of $P$ of order $q$, then $x$ acts irreducibly on $P/Z(P)$.*

*Proof.* By Burnside's Basis Theorem $x$ acts nontrivially on $\overline{P} = P/Z(P)$. Suppose that the $\mathbb{F}_p\langle\, x\,\rangle$-module $\overline{P}$ is not irreducible. Since $\mathrm{char}(\mathbb{F}_p) = p$ does not divide $|\,\langle\, x\,\rangle\,| = q$, by Maschke's Theorem $\overline{P}$ is a direct sum of $\langle\, x\,\rangle$-invariant subspaces:

$$\overline{P} = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \qquad k \geq 2.$$

Since $x$ acts nontrivially on $\overline{P}$, it acts nontrivially on, say, $V_1$. Thus $q = |\, x\,|$ divides $|\, GL(V_1)\,|$, which is a product of factors $p^d - p^i$ where $d = \dim V_1$ and $i < d$. This contradicts the choice of the Zsigmondy prime $q$. □

Since $Q_1$ is a $q$-group its center is nontrivial, so take $x \in Z(Q_1)$ of order $q$. Then $C_Q(x)$ acts faithfully on $\overline{P} = P/Z(P)$ by Theorem 3.2.3. Since $C_Q(x)$ commutes with the irreducible action of $x$ on $\overline{P}$, by Schur's Lemma $C_Q(x)$ is cyclic and has order prime to $p$ (we note that Schur's Lemma applies in any characteristic). In particular, $Q_1$ is cyclic. Since $Q/C_Q(Q_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q_1)$, $Q/C_Q(Q_1)$ is abelian and has order dividing $\phi(|\, Q_1\,|)$, where $\phi$ is the Euler totient function. Since by hypothesis $p \nmid q - 1$, we have shown that

*$p$ does not divide $|\, Q\,|$ and $Q$ is solvable.*

Note that since $P$ and $Q$ are both solvable, $L = P \rtimes Q$ is as well. In Chapter 4 we shall prove that $L$ is not a subnormal subgroup of any M-group. In this section we first show the following:

**Theorem 3.4.2.** *Let $p$ be an odd prime, let $P$ be an extraspecial $p$-group of order $p^{1+2m}$ and exponent $p$, and let $Q \leq \mathrm{Aut}(P)$ satisfy:*

*(a)* $p \nmid |\, Q\,|$.

*(b) A prime divisor $q$ of $|\, Q\,|$ is a Zsigmondy prime for $(p, 2m)$.*

*(c) $Q$ is a self-normalizing subgroup of* $\mathrm{Aut}(P)$*, that is,* $N_{\mathrm{Aut}(P)}(Q) = Q$.

*(d) $Q$ acts nontrivially on $Z(P)$.*

*(e) $Q$ is solvable.*

*Then $L = P \rtimes Q$ is a complete non-M-group.*

Note that we have explicitly constructed a family of solvable groups that satisfy the hypotheses of Theorem 3.4.2.

**Lemma 3.4.3.** *Let $L$ be as in Theorem 3.4.2. Then $Z(P)$ is the unique minimal normal subgroup of $L$.*

*Proof.* Let $N \trianglelefteq L$. If $p$ divides $|\,N\,|$ then $1 < N \cap P \trianglelefteq P$, so $Z(P) \leq N$ by Lemma 3.2.2. Assume instead that $p \nmid |\,N\,|$. Then by Hall's Theorem, $N \leq Q$. But now $[P, N] \leq P \cap N = 1$, so $N$ centralizes $P$. Since $Q$ acts faithfully on $P$, we must have $N = 1$. □

**Lemma 3.4.4.** *Any group $L$ that satisfies the hypotheses of Theorem 3.4.2 is complete.*

*Proof.* Since $Q$ acts nontrivially by conjugation on $Z = Z(P)$ by assumption we have $Z \not\leq Z(L)$, hence $Z(L) = 1$ by Lemma 3.4.3. Let $G = \mathrm{Aut}(L)$ and identify $L$ with the normal subgroup $\mathrm{Inn}(L)$ of $G$.

Since $P = O_p(L)$ is characteristic in $L$, $P \trianglelefteq G$. Then $G$ acts on $P$ by conjugation with kernel $C = C_G(P)$. Let $\alpha \in C$ be nontrivial. Then for any $x \in L$,

$$[x, \alpha] \in [L, C] \leq L \cap C = Z.$$

Hence $x^\alpha = xz$ for some $z \in Z$. Since $\alpha$ centralizes $Z$, one easily computes that $x^{\alpha^i} = xz^i$ for all $i$. When $i = p$ we have $x^{\alpha^p} = x$ and since $x \in L$ was chosen arbitrarily, we see that $|\,\alpha\,| = p$. So $C$ is a $p$-group on which the $p'$-group $Q$ acts. By Lemma 2.1.7,

$$C = [Q, C] \cdot C_C(Q) = Z \times C_C(Q).$$

But $C_C(Q) \leq C_G(L) = 1$. Thus $C = Z$.

By what we have just shown $\overline{G} = G/Z$ is (isomorphic to) a subgroup of $A = \mathrm{Aut}(P)$. Since $Q$ is a Hall $p'$-subgroup of the solvable group $\overline{L}$ and all such subgroups are conjugate in $\overline{L}$, Frattini's argument yields $\overline{G} = N_{\overline{G}}(Q)\overline{L}$. But $N_A(Q) = Q$ by assumption, so

$$\overline{G} = N_{\overline{G}}(Q)\overline{L} = (N_A(Q) \cap \overline{G})\overline{L} = Q\overline{L} = \overline{L},$$

and it follows that $G = L$, completing the proof. □

Theorem 3.4.2 is established once we prove the following result.

**Lemma 3.4.5.** *The group $L$ of Theorem 3.4.2 is a non-M-group.*

*Proof.* By Theorem 3.2.4, $P$ has a faithful irreducible character $\varphi$ of degree $p^m$ that is uniquely determined by its value on a generator for $Z = Z(P)$. Let $T$ denote the inertia group of $\varphi$ in $L$; so $T = C_L(Z)$. Since $P$ is a normal Hall subgroup of $T$, by Theorems 2.2.3 and 2.2.5 there exists an irreducible character $\zeta$ of $T$ such that $\zeta_P = \varphi$ and $\zeta^L = \psi \in \mathrm{Irr}(L)$. Note that $\psi(1) = p^m|\, L : T\,|$. Since $L/T$ is isomorphic to a subgroup of $\mathrm{Aut}(Z)$, the index $|\, L : T\,|$ divides $|\, \mathrm{Aut}(Z)\,| = p - 1$. It follows that $q \nmid \psi(1)$.

If $\psi$ is monomial, then there exists a subgroup $H$ of $L$ with $|L : H| = \psi(1)$. A Sylow $q$-subgroup of $L$ is contained in $H$ by the previous paragraph, hence $H$ acts irreducibly on $\overline{P} = P/Z$ by Lemma 3.4.1. Now the $p$-part of $|\, L\,|$ is $p^{1+2m} = |\, P\,|$ and the $p$-part of $|\, L : H\,|$ is $p^m = \varphi(1)$, and thus a Sylow $p$-subgroup of $H$ has order $p^{1+m}$. Set $R = H \cap P \in \mathrm{Syl}_p(H)$. Then by order considerations $R \not\leq Z$. Since $R \trianglelefteq H$, $\overline{R}$ is a nonzero $H$-stable subspace of $\overline{P}$. By irreducibility $\overline{R} = \overline{P}$; that is, $RZ = P$. But $Z = \Phi(P)$, the Frattini subgroup of $P$, so by Burnside's Basis Theorem we have $R = P$, a contradiction. We conclude that $\psi$ is nonmonomial and that $L$ is a non-M-group. □

# Chapter 4

# Subgroups of M-Groups

A well-known result due to Dade asserts that every solvable group embeds in an M-group. We begin this chapter with a new proof of Dade's Embedding Theorem. We then show that a certain refinement to this theorem cannot be made without the addition of any new hypotheses. Specifically, we will show that there are solvable groups which cannot be embedded subnormally in any M-group.

## 4.1 Dade's Embedding Theorem

Dade's result is a corollary of the following theorem. Although this appears in the literature — for example, as Theorem 18.10 of [16] — we give a new, shorter proof.

**Theorem 4.1.1.** *Let $H$ be an M-group and let $Z$ be a cyclic group of prime order $p$. Then $G = H \wr Z$ is an M-group.*

*Proof.* Let $H_0$ denote the direct product of $p$ copies of $H$, so that $G = H_0 \rtimes Z$. Note that $H_0$ is an M-group by Lemma 2.2.12. If $\chi$ is an irreducible character of $G$ then by Corollary 2.2.4 the restriction of $\chi$ to $H_0$ is either irreducible or the sum of $p$ distinct irreducible characters. In the latter case $\chi$ is induced from any of the irreducible constituents of $\chi_{H_0}$, and hence is monomial by transitivity of induction. We may assume, therefore, that $\chi_{H_0}$ is irreducible. In this case,

$$\chi_{H_0} = \underbrace{\varphi \otimes \varphi \otimes \cdots \otimes \varphi}_{p}$$

for some $\varphi \in \operatorname{Irr}(H)$. Set $\varphi_0 = \chi_{H_0}$. Now, since $H$ is an M-group, there exists a linear character $\lambda$ of a subgroup $K$ of $H$ such that $\lambda^H = \varphi$. Let

$$K_0 = \underbrace{K \times K \times \cdots \times K}_{p} \leq H_0 \qquad \text{and} \qquad \lambda_0 = \underbrace{\lambda \otimes \lambda \otimes \cdots \otimes \lambda}_{p}.$$

Then $\lambda_0$ is a linear character of $K_0$ and $\lambda_0^{H_0} = \varphi_0$.

Since $Z$ normalizes $K_0$, we may form the subgroup $M = K_0 Z$. Then by Corollary 2.2.4, $\lambda_0$ extends to a linear character, say, $\mu$ of $M$ since it is invariant under $Z$.

We compute $\lambda_0^G$ in two ways. First, we have

$$\lambda_0^G = (\lambda_0^{H_0})^G = \varphi_0^G = \rho_{G/H_0}\chi$$

by Theorem 2.2.6. On the other hand,

$$\lambda_0^G = (\lambda_0^M)^G = (\rho_{M/K_0}\mu)^G = \rho_{G/H_0}\mu^G,$$

again by Theorem 2.2.6 but also by Lemma 2.2.1. Thus, for some linear $\beta \in \operatorname{Irr}(G/H_0)$ we have $\chi = \beta\mu^G = (\beta_M\mu)^G$, completing the proof. □

**Corollary 4.1.2.** *(Dade) Every solvable group is a subgroup of an M-group.*

*Proof.* Let $H$ be any solvable group. Then there exists a series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r = H$$

such that each factor $H_i/H_{i-1}$ is cyclic of prime order (this is a *composition series* for $H$).

Say $|\,H_i/H_{i-1}\,| = p_i$ where the $p_i$'s are not necessarily distinct. Then by Theorem 2.1.4 each $H_i$, $i \geq 1$, embeds in the wreath product $H_{i-1} \wr (H_i/H_{i-1}) \cong H_{i-1} \wr Z_{p_i}$. It follows that $H$ embeds as a subgroup of the iterated wreath product

$$G = (((Z_{p_1} \wr Z_{p_2}) \wr Z_{p_3}) \wr \cdots) \wr Z_{p_r}.$$

The group $G$ is an M-group by repeated applications of Theorem 4.1.1. □

It is worth noting that iterated wreath products, such as those constructed in the proof of Corollary 4.1.2, appear naturally in many contexts; for example, as Sylow subgroups of symmetric groups.

## 4.2 Subnormal Subgroups of M-Groups

It is amusing to observe how unwieldy the groups $G$ constructed in the proof of Corollary 4.1.2 can become, even when the subgroup $H$ has small order. For example, when $H = SL(2,3)$, the smallest solvable non-M-group, the corresponding group[1] $G$ has order $6,291,456 = 2^{21} \cdot 3$. If instead we begin with $H = GL(2,3)$ then the order of $G$ is $79,164,837,199,872 = 2^{43} \cdot 3^2$.

On the other hand, we need not follow the proof of Corollary 4.1.2 to the letter in order to embed a given solvable group in an M-group. For instance, $SL(2,3)$ is a subgroup of $Q_8 \wr Z_3$ by the Universal Embedding Theorem for Wreath Products, and the latter group is an M-group by Theorem 4.1.1.[2] The wreath product in this case has order $1,536 = 2^9 \cdot 3$.

[1] There may be many different iterated wreath products $G$ that we can embed a general solvable group $H$ in. However, $SL(2,3)$ and $GL(2,3)$ each have unique composition series. Thus we may speak of *the* corresponding group $G$.

[2] In fact, $SL(2,3)$ is a subgroup of an even smaller M-group. The embedding of $SL(2,3)$ in $W = Q_8 \wr Z_3$ maps the center of $SL(2,3)$ into $Z(W)$. Let $B = Q_1 \times Q_2 \times Q_3$ denote the base group of $W$, and write $z_i$ for the generator of $Z(Q_i)$. Then the subgroup $Z_0 = \langle\, z_1z_2, z_2z_3, z_1z_3 \,\rangle$ is normal in $W$ and a complement to $\langle\, z_1z_2z_3 \,\rangle$ in the center of $Q_1Q_2Q_3$. Hence $SL(2,3)$ embeds in $W/Z_0 \cong (Q_1 * Q_2 * Q_3) \rtimes Z_3$, and the latter group is an M-group since it is a quotient of $W$. Even better, it can be shown that $SL(2,3)$ is a subgroup of index 4 in the semidirect product $(Q_8 * Q_8) \rtimes Z_3$, which is also an M-group.

It seems natural to wonder if Corollary 4.1.2 can be strengthened in some way. One might ask if any solvable group can be embedded as a *normal* subgroup of some M-group. The answer — no — is an immediate consequence of earlier results.

**Theorem 4.2.1.** *Let $H$ be a complete non-M-group. Then $H$ is not a normal subgroup of any M-group.*

*Proof.* If $G$ contains $H$ as a normal subgroup, then by Lemma 2.1.3 the group $G$ decomposes into the direct product $H \times C_G(H)$. Thus $G$ is not an M-group by Lemma 2.2.12. □

Examples of complete non-M-groups were given in Sections 3.3 and 3.4.

So, given a solvable group $H$, it is in general too much to ask for an M-group $G$ containing $H$ normally. We ask instead whether or not $H$ can be embedded *subnormally* in an M-group. Our main theorem is that the answer to this question is "no."

**Theorem 4.2.2.** *Let $p$ be an odd prime, let $P$ be an extraspecial $p$-group of order $p^{1+2m}$ and exponent $p$, and let $Q \leq \mathrm{Aut}(P)$ satisfy:*

*(a)* $p \nmid |\,Q\,|$.

*(b) A prime divisor $q$ of $|\,Q\,|$ is a Zsigmondy prime for $(p, 2m)$.*

*(c) $Q$ is a self-normalizing subgroup of $\mathrm{Aut}(P)$, that is, $N_{\mathrm{Aut}(P)}(Q) = Q$.*

*(d) $Q$ acts nontrivially on $Z(P)$.*

*Then $L = P \rtimes Q$ is not a subnormal subgroup of any M-group.*

If the group $L$ of Theorem 4.2.2 is not solvable then $L$ is trivially not a subgroup of an M-group by Theorem 2.2.9. Note that solvable groups $L$ were exhibited in Section 3.4. Note also that by condition (b) and Burnside's Basis Theorem, any element of order $q$ in $Q$ — hence a fortiori $Q$ itself — acts irreducibly and faithfully on $P/Z(P)$.

**Lemma 4.2.3.** *Let $L$ be a solvable group that satisfies the hypotheses of Theorem 4.2.2. Let $L_1 \cong L_2 \cong L$ and write $L_i = P_i \rtimes Q_i$ for $i = 1, 2$. Assume that $L_1$ and $L_2$ are both normal subgroups of a group $G$ that satisfies $O_{p'}(G) = 1$. If $L_1L_2 \neq L_1 \times L_2$, then $L_1 = L_2$.*

*Proof.* We may assume without loss of generality that $G = L_1L_2$, since $O_{p'}(L_1L_2) \leq O_{p'}(G) = 1$. Note that $G = L_1 \times C_G(L_1)$ by Lemma 2.1.3. Since $Z_1$ is the unique minimal normal subgroup of $L_1$ by Lemma 3.4.3, $Z_1$ is characteristic in $L_1$ and hence normal in $G$. We write $\overline{G}$ for the quotient $G/Z_1$.

Suppose that $[L_1, L_2] \leq Z_1$. Then $[L_1, Q_2] \leq Z_1$, so $C_{\overline{L_1}}(Q_2) = \overline{L_1}$. Now $|\, Z_1 \,|$ and $|\, Q_2 \,|$ are relatively prime so $C_{\overline{L_1}}(Q_2) = \overline{C_{L_1}(Q_2)}$ and it follows that $|\, L_1 : C_{L_1}(Q_2) \,| = 1$ or $p$. The latter is impossible since $L_1$ has no subgroup of index $p$, hence $C_{L_1}(Q_2) = L_1$ or what is the same, $[L_1, Q_2] = 1$. In fact for any $p'$-Hall subgroup $Q^*$ of $L_2$ we have $[L_1, Q^*] = 1$, and since $L_2$ is generated by its $p'$-Hall subgroups the equality $[L_1, L_2] = 1$ follows. In other words $L_2 \leq C_G(L_1)$, so that $G = L_1 \times L_2$, a contradiction.

It must be the case, then, that $[L_1, L_2] \not\leq Z_1$. We show that $P_1 = P_2 \leq [L_1, L_2]$. Set $D = [L_1, L_2]$ and note that $D \trianglelefteq L_1$. Since $Z_1$ is the unique minimal normal subgroup of $L_1$ we must have $Z_1 < D$. Now, if $Z_1$ is a Sylow $p$-subgroup of $D$ then we may write $D = Z_1 \rtimes Q_0$ for some $Q_0 \leq Q_1$. But then $Q_0 \trianglelefteq \overline{P_1} \rtimes Q_0$, so the semidirect product action is trivial, contradicting Burnside's Basis Theorem. Hence a Sylow $p$-subgroup of $D$ properly contains $Z_1$, i.e., $D \cap P_1 > Z_1$. Moreover $P_1 \leq D$ by the irreducible action of $Q_1$. By symmetry $P_2 \leq D$ as well, and since $P_1$ is a normal Sylow $p$-subgroup of $D$ we see that $P_1 = P_2$, as needed.

So a Sylow $p$-subgroup of $G$ is contained in $L_1$, and in particular $G/L_1 \cong C_G(L_1)$ is a $p'$-group. Since $C_G(L_1) \trianglelefteq G$ and by hypothesis $O_{p'}(G) = 1$, we see that $C_G(L_1) = 1$. That is, $G = L_1$, and the proof is complete. $\square$

**Lemma 4.2.4.** *Let $L$ be a solvable group that satisfies the hypotheses of Theorem 4.2.2, and assume that $L \trianglelefteq\trianglelefteq G$ and $O_{p'}(G) = 1$. Write $L^G = \{L_1, L_2, \ldots, L_s\}$ for the set of* *distinct conjugates of $L$ in $G$ and $L_i = P_i \rtimes Q_i$ for $1 \leq i \leq s$. Then*

$$\langle L^G \rangle = L_1 \times L_2 \times \cdots \times L_s.$$

The group $\langle L^G \rangle$ of Lemma 4.2.4 is called the *normal closure of $L$ in $G$.*

*Proof.* If $L \trianglelefteq G$ then the proof is trivial, so assume instead that $L$ is not normal in $G$. Then there is a proper normal subgroup $M$ of $G$ that contains $L$ subnormally:

$$L \trianglelefteq\trianglelefteq M \triangleleft G.$$

Note that $\langle L^G \rangle \leq M$. Relabeling if necessary, we may write $L^M = \{L_1, L_2, \ldots, L_r\}$ for the set of distinct conjugates of $L$ in $M$. Now $O_{p'}(M) \leq O_{p'}(G)$ since $M \trianglelefteq G$, hence $O_{p'}(M) = 1$. By induction,

$$N = \langle L^M \rangle = L_1 \times L_2 \times \cdots \times L_r.$$

Note that $N \leq \langle L^G \rangle$ since $L^M \subseteq L^G$. We may assume that these containments are proper, for otherwise, the proof is complete. Let $L_k \in L^G - L^M$. We show, by way of contradiction, that $L_k$ normalizes each $L_i \in L^M$. Suppose that $L_k$ does not normalize, say, $L_1$. Then $L_1^x \neq L_1$ for some $x \in Q_k$, and without loss of generality we may assume that $L_1^x = L_2$. Now $P_1 \leq O_p(N)$ and $O_p(N) = O_p(G) \cap N$ since $N \trianglelefteq\trianglelefteq G$, hence

$$[P_1, Q_k] \leq [O_p(G), Q_k].$$

Since $L_k$ is subnormal in the solvable subgroup $O_p(G)L_k$ of $G$, there is a subnormal series

$$L_k = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = O_p(G)L_k$$

such that $|\,H_{i+1} : H_i\,| = p$ for all $i$. We claim that $[O_p(G), Q_k] \leq O_p(H_i)$ for all $i$. Note first that $[O_p(G), Q_k] \leq O_p(G) \leq O_p(H_n)$. Recall that $[O_p(G), Q_k] = [O_p(G), Q_k, Q_k]$ by Lemma 2.1.7. By induction,

$$[O_p(G), Q_k] = [O_p(G), \underbrace{Q_k, Q_k, \ldots, Q_k}_{n-i+1}] \leq [O_p(H_{i+1}), Q_k].$$

Now, $[O_p(H_{i+1}), Q_k] \leq O_p(H_{i+1})$ since $Q_k$ normalizes $O_p(H_{i+1})$ and $[O_p(H_{i+1}), Q_k] \leq [H_{i+1}, H_{i+1}] \leq H_i$ because $H_{i+1}/H_i \cong Z_p$. Consequently

$$[O_p(G), Q_k] \leq [O_p(H_{i+1}), Q_k] \leq O_p(H_{i+1}) \cap H_i = O_p(H_i),$$

which verifies the claim. Note that when $i = 0$ we have

$$[P_1, Q_k] \leq [O_p(G), Q_k] \leq O_p(H_0) = P_k.$$

Now $[P_1, Q_k] \geq [P_1, x]$ so $[P_1, Q_k]$ contains a "diagonal" subgroup of $P_1 \times P_1^x$ isomorphic to $P_1$. But also

$$[P_1, Q_k] \supseteq \{[y, x] : y \in P_1\} \supseteq P_1' = Z(P).$$

Therefore

$$|\,P_1\,| < |\,[P_1, Q_k]\,| \leq |\,P_k\,|,$$

which is a contradiction. We conclude that $L_k$ normalizes each $L_i \in L^M$. Since $L_k$ was arbitrary we may apply Lemma 4.2.3 to obtain

$$\langle\, L^G \,\rangle = L_1 \times L_2 \times \cdots \times L_s. \qquad \square$$

*Proof of Theorem 4.2.2.* Suppose that there exists an M-group $G$ containing $L$ as a subnormal subgroup. Then $L$ must be solvable, so $L \cap O_{p'}(G) = O_{p'}(L) = 1$ by Lemma 3.4.3. In particular, $L$ embeds as a subnormal subgroup of $G/O_{p'}(G)$. The quotient group $G/O_{p'}(G)$ is an M-group by Lemma 2.2.11, so we may assume that $O_{p'}(G) = 1$.

Let $L^G = \{L_1, L_2, \ldots, L_s\}$ be the set of distinct conjugates of $L$ in $G$. By Lemma 4.2.4,

$$N = \langle\, L^G \,\rangle = L_1 \times L_2 \times \cdots \times L_s.$$

Let $\varphi \in \mathrm{Irr}(P)$ be faithful of degree $p^m$. Since $\varphi$ is uniquely determined by its value on $Z = Z(P)$, we have $T = I_L(\varphi) = C_Q(Z)P$. By Theorems 2.2.3 and 2.2.5, there exists a character $\zeta \in \mathrm{Irr}(T)$ such that $\zeta_P = \varphi$ and $\zeta^L = \psi \in \mathrm{Irr}(L)$. Note that $\ker\psi = 1$; otherwise, the containments $Z \leq \ker\psi \leq \ker\zeta$ would hold, contradicting the fact that $\varphi$ is faithful. Note also that $\psi(1) = \zeta(1)|\, L : T\,| = \varphi(1)|\, Q : C_Q(Z)\,|$, so $q \nmid \psi(1)$.

The character

$$\theta = \underbrace{\psi \otimes \psi \otimes \cdots \otimes \psi}_{s} \in \mathrm{Irr}(N)$$

extends to a character $\chi$ of $G$ by Theorem 2.2.18, and $\theta$ is faithful by the remarks following Lemma 2.2.1. Since $G$ is an M-group, there exists a subgroup $H \leq G$ and a linear character $\lambda$ of $H$ such that $\lambda^G = \chi$. Then $q$ does not divide $|\, G : H\,| = \psi(1)^s$, so $H$ contains a Sylow $q$-subgroup of $G$.

For each $i \in \{1, 2, \ldots, s\}$, let $P_i$ be the Sylow $p$-subgroup of $L_i$. Write $\overline{P_i} = P_i/Z(P_i)$ and let

$$V = \overline{P_1} \times \overline{P_2} \times \cdots \times \overline{P_s}.$$

Let $Q_0$ be a Sylow $q$-subgroup of $H \cap N$. Since an element of order $q$ in $L_i$ acts irreducibly on $\overline{P_i}$, each $\overline{P_i}$ is an irreducible $\mathbb{F}_pQ_0$-submodule of $V$. Furthermore, the $\mathbb{F}_pQ_0$-submodules $\overline{P_i}$ are pairwise nonisomorphic since they have different kernels. Thus any nonzero $Q_0$-stable subspace of $V$ is a direct product of some subset of $\left\{\overline{P_i} : 1 \leq i \leq s\right\}$.

Now $(\lambda^G)_N = \theta$, so $G = NH$ by Mackey's Theorem. Since the $p$-part of $|\, G : H\,|$ is $\varphi(1)^s = p^{ms}$ and $p$ does not divide $|\, G : O_p(N)H\,| = |\, N : O_p(N)\,|$, we must have

$|\, O_p(N)H : H\,| = p^{ms}$. Let $R = H \cap O_p(N)$ and let $Z_0 = Z(O_p(N))$. Then

$$|\, R\,| = \frac{|\, O_p(N)\,|}{|\, O_p(N) : R\,|} = \frac{p^{(1+2m)s}}{p^{ms}} = p^{(1+m)s}.$$

By order considerations, $R \not\leq Z_0$. Note that $R \trianglelefteq H$, so $Q_0$ normalizes $R$. In particular, $RZ_0/Z_0$ is a nonzero $Q_0$-stable subspace of $V$. By the previous paragraph and Burnside's Basis Theorem we must have $P_i \leq H$ for some $i$. Thus

$$Z(P_i) = P_i' \leq H' \leq \ker \lambda.$$

By Theorem 2.2.16, $\theta = (\lambda_{H\cap N})^N$. So then

$$1 = \ker \theta = \bigcap_{n\in N} (\ker \lambda_{H\cap N})^n = \bigcap_{n\in N} (N \cap \ker \lambda)^n \geq Z(P_i).$$

This contradiction completes the proof.

$\square$

# Chapter 5

# Almost M-Groups

As observed in Section 3.1, the linear group $SL(2,3)$ has exactly 3 nonmonomial irreducible characters, while $GL(2,3)$ has 2. In this chapter we study groups possessing only 1 nonmonomial irreducible character. We show that such a group cannot be simple using the Classification of Finite Simple Groups.[1]

**Definition 5.0.5.** *A group $G$ is an* almost M-group *if it has a unique nonmonomial irreducible character.*

**Theorem 5.0.6.** *Almost M-groups are not simple.*

*Proof.* The proof is by contradiction. Let $G$ be a simple almost M-group and let $\phi$ denote the unique nonmonomial character in $\mathrm{Irr}(G)$. Note that the principal character $1_G$ is the only linear character of $G$. Let $f$ be the smallest degree of a nonprincipal irreducible character of $G$ and suppose that for some monomial $\chi \in \mathrm{Irr}(G)$ we have $\chi(1) = f$. Then there must exist a subgroup $K < G$ of index $f$. But by minimality of $f$ the only irreducible constituent of $(1_k)^G$ is $1_G$, hence $K = G$, a contradiction. It follows that $\phi(1) = f$. Furthermore, $\phi$ is the unique irreducible character of $G$ of degree $f$.

[1] It turns out that almost M-groups are necessarily solvable, since Qian has shown that if all nonmonomial irreducible characters of $G$ have the same degree, then $G$ is solvable [21]. His proof also relies on the Classification heavily. Our proof is independent of the arguments of Qian.

Choose $\chi \in \operatorname{Irr}(G)$ subject to the constraint

$$\chi(1) = \min\{\psi(1) : \psi \in \operatorname{Irr}(G) \text{ is monomial and } \psi \neq 1_G\}.$$

Then $\chi$ is monomial, and there is a (nonprincipal) linear character of a subgroup $H < G$ that induces $\chi$. Write $(1_H)^G = 1_G + \psi$ where $\psi$ is a possibly reducible character of $G$. Since $\psi(1) < |\,G : H\,| = \chi(1)$ and $(\psi, 1_G) = 0$, we see that $\phi$ is the sole irreducible constituent of $\psi$. Thus we may write

$$(1_H)^G = 1_G + n\phi \tag{$*$}$$

for some $n \in \mathbb{N}$. No group is equal to the union of all conjugates of a proper subgroup, so there exists an element $g \in G$ such that $g$ is not contained in any conjugate of $H$. By the formula for induced characters $(1_H)^G(g) = 0$, so that we have

$$\phi(g) = -\frac{1}{n}.$$

But $\phi(g)$ is an algebraic integer, so $n = 1$. Hence by basic permutation group theory $(*)$ now implies that $G$ acts 2-transitively on the set of left cosets of $H$ in $G$.

If $H_0 < G$ with $|\,G : H_0\,| < |\,G : H\,|$, then $(1_{H_0})^G = 1_G + \omega$ for some character $\omega$ of $G$. But now $\omega(1) < |\,G : H_0\,| < |\,G : H\,| = 1 + f$, so that $\omega(1) < \phi(1)$. This forces $(\omega, 1_G) \neq 0$, a contradiction.

In summary, if $G$ is a simple almost M-group, then:

(a) A proper subgroup $H$ of $G$ of smallest possible index satisfies $H' < H$.

(b) $G$ acts 2-transitively on the set of left cosets $G/H$.

(c) $|\,G : H\,| = 1 + \phi(1)$.

(d) If $\phi \neq \chi \in \operatorname{Irr}(G)$ is such that $\chi(1) \leq \phi(1)$ then $\chi = 1_G$.

Every 2-transitive simple group is listed in Table 5.1. That this list is complete is established in [3] (the result relies ultimately on the Classification of the Finite Simple Groups). In particular, $G$ must belong in this table.

| Group | Degree | Transitivity | Notes |
|---|---|---|---|
| $A_n$ | $n$ | $n-2$ | $n \geq 5$ |
| $PSL(2,q)$ | $(q^2-1)/(q-1)$ | 3 | $q \neq 2,3$ |
| $PSL(n,q)$ | $(q^n-1)/(q-1)$ | 2 | $n \geq 2$ |
| $PSU(3,q)$ | $q^3+1$ | 2 | $q > 2$ |
| ${}^2B_2(q)$ | $q^2+1$ | 2 | $q = 2^{2a+1}$, $a \geq 1$ |
| ${}^2G_2(q)$ | $q^3+1$ | 2 | $q = 3^{2a+1}$, $a \geq 1$ |
| $PSp(2d,2)$ | $2^{2d-1} \pm 2^{d-1}$ | 2 | $d > 2$ |
| $PSL(2,11)$ | 11 | 2 | |
| $PSL(2,8)$ | 28 | 2 | |
| $A_7$ | 15 | 2 | |
| $M_{11}$ | 11 | 4 | |
| $M_{11}$ | 12 | 3 | |
| $M_{12}$ | 12 | 5 | |
| $M_{22}$ | 22 | 3 | |
| $M_{23}$ | 23 | 4 | |
| $M_{24}$ | 24 | 5 | |
| $HS$ | 176 | 2 | |
| $Co_3$ | 276 | 2 | |

*Table 5.1: 2-Transitive Simple Groups.*

Character tables of most of these groups can be found in the *ATLAS of Finite Groups* [4], which we rely on in what follows without further reference.

If $G$ were alternating, then condition (a) would force $G \cong A_5$. But $A_5$ has two irreducible characters of degree 3 and no subgroup of index 3, so is not an almost M-group.

Suppose $G \cong PSL(2,q)$. Then by (c) we must have $\phi(1) = q$. But for all $q$, $G$ has an irreducible character of degree $q-1$, contrary to (d); see Chapter 38 of [8].

Suppose $G \cong PSL(n,q)$ with $n \geq 3$. Then by (b) and (c) we have

$$\phi(1) = \frac{q^n-1}{q-1} - 1.$$

The (simple) projective special linear groups $PSL(n,q)$ always have an irreducible character of $q$-power degree known as the *Steinberg character*. This character is not monomial, with the two exceptions of $A_5$ and $PSL(2,7)$ (see [19]), and we have seen already that neither of these is an almost M-group. Thus if $PSL(n,q)$, $n \geq 3$, is an almost M-group we must have that $\phi$ is the Steinberg character and so $\phi(1)$ a power of $q$. But elementary number theory shows that $\phi(1)$ cannot be a power of $q$, a contradiction.

We cannot have $G \cong PSU(3,q)$: if we did, then $\phi(1) = q^3$ by (c); but $PSU(3,q)$ has an irreducible character of degree $q(q-1)$ as established in [12], contrary to (d).

Suppose that $G$ is the Suzuki group ${}^2B_2(q)$ for some $q = 2^{2a+1}$, $a \geq 1$. Then (c) forces $\phi(1) = q^2$. But in [24] Suzuki proved that ${}^2B_2(q)$ has nonprincipal irreducible characters of degree strictly less than $q^2$, again contradicting (d). Thus ${}^2B_2(q)$ cannot be an almost M-group.

The Ree groups ${}^2G_2(q)$ for $q = 3^{2a+1}$, $a \geq 1$, have irreducible characters of degree $q^2 - q + 1$ (see the character table given in [27]). Since $q^2 - q + 1 < q^3$, ${}^2G_2(q)$ is not an almost M-group by (d).

The projective symplectic groups $PSp(2d,2)$ for $d > 2$ have a Steinberg character of 2-power degree that is nonmonomial except possibly when $d = 4$; see [19]. By the Atlas, $PSp(8,2)$ has irreducible characters of degree 35 and 51 but no subgroups of these indices, so $PSp(8,2)$ is not an almost M-group. Thus if $G \cong PSp(2d,d)$ then $\phi(1)$ must be a power of 2. But we know by Table 5.1 and (c) that

$$\phi(1) = 2^{2d-1} \pm 2^{d-1} - 1$$

which is odd, a contradiction.

Since the Mathieu group $M_{11}$ has 3 irreducible characters of degree 10, and no nonprincipal irreducible characters of degree less than 10, $M_{11}$ is not an almost M-group by (d). The same argument shows that $M_{12}$ is not an almost M-group, either.

The groups $M_{22}$, $M_{23}$, and $M_{24}$ each have 2 irreducible characters of degree 45, but none have subgroups of index 45.

The Higman-Sims group $HS$ has irreducible characters of degree 22 and 77, while the smallest index of a proper subgroup is 100, contradicting (c) and (d).

Finally, if $G$ is isomorphic to the Conway group $Co_3$, then $|\,G : H\,| = 276$. However $G$ has an irreducible character of degree 23, which goes against (c).

This eliminates all groups in Table 5.1, and so verifies that simple almost M-groups do not exist. $\square$

# Chapter 6

# Further Questions

Our research into this area of character theory produced a number of questions that we have yet to answer. This thesis concludes with a discussion of some of these possible avenues for future investigation.

- Can $SL(2,3)$, the smallest non-M-group, be embedded as a subnormal subgroup of some M-group?

The groups constructed in Chapter 3 and $SL(2,3)$ share the same "$P \rtimes Q$" form. However, the property of being complete was essential to our construction. Since the center of $SL(2,3)$ is nontrivial, $SL(2,3)$ is not complete.

More generally, we would like to know:

- What properties do normal or subnormal subgroups of M-groups have? Can they be characterized in some fashion?

- Is it possible to classify the maximal subgroups of M-groups?

- What can one say about the family of groups $G$ with the property that every irreducible character is induced from a linear character of some *subnormal* subgroup of $G$? This family includes all nilpotent groups, since every subgroup of a nilpotent group is subnormal. It also includes Frobenius groups with cyclic Frobenius complements (which need not be supersolvable).

Of course, a normal subgroup of an M-group need not be an M-group. However, Isaacs has conjectured that normal subgroups of odd order M-groups are M-groups, and Lewis has made some progress in proving this [20].

- If $\varphi$ and $\psi$ are nonmonomial characters of groups $A$ and $B$, respectively, is the tensor product $\varphi \otimes \psi$ a nonmonomial character of $A \times B$? More generally, can one characterize the nonmonomial irreducible characters of $A \times B$?

These questions seem answerable. A proof of the former eluded us, though we did not make a serious effort. We note that if the tensor product of nonmonomial characters is nonmonomial, the proof of Theorem 4.2.2 could be simplified considerably.

- Can one prove that almost M-groups are not simple, or, better yet, solvable, without relying on the Classification of the Finite Simple Groups?

- For a fixed positive integer $N$, can one find a bound on $|\,G\,|$ for $G$ any nonsolvable group having no more than $N$ nonmonomial irreducible characters?

One may be able to answer the latter question for the class of simple groups by applying the Classification of the Finite Simple Groups.

- Do almost M-groups exist?

In Section 3.1 we noted that $GL(2,3)$ has exactly 2 nonmonomial irreducible characters, and we saw that $SL(2,3)$ has 3. To date, we have yet to find a (solvable) group with exactly 1 nonmonomial irreducible character, and we suspect that they do not exist. We remark that if $G$ is an almost M-group with nonmonomial irreducible character $\phi$ then $|\,G\,|$ cannot be odd, since a well-known result of Burnside states that the nonprincipal irreducible characters of a group of odd order are never fixed under complex conjugation, and $\overline{\phi}$ is monomial if and only if $\phi$ is.

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