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THE CLASSICAL AND QUANTUM LAMB MODEL

A Thesis Presented

by

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ABSTRACT

In 1900, Horace Lamb considered the classical dynamics of a vibrating particle embedded in an elastic medium. Lamb described how the back action of the elastic waves generated damps the vibrations of the particle. This process is of renewed interest in the quantum regime. The quantum theory of such dissipative systems finds relevance to a variety of mesoscopic and nanoscopic mechanical systems currently under intense experimental study. The Lamb model is revisited here using a Lagrangian framework. The coupling between the vibrating particle and the elastic medium arises from enforcing a holonomic constraint with a Lagrange multiplier. The classical equations of motion, obtained from the Lagrangian, are subsequently solved exactly using integral transform methods. The model is then quantized, and the acoustic power radiated is calculated using time-dependent perturbation theory. These results are valid in the regime of light damping.

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CHAPTER 1

INTRODUCTION

The harmonic oscillator has been described as the “most important” physical system in all of physics for mainly two reasons: (i) the undamped harmonic oscillator is an *exactly* solvable system in both classical and quantum mechanics [1,2], and (ii) most objects behave like a simple harmonic oscillator if their oscillations (in magnitude) are small enough. By exactly solvable, what I mean is that one can write down the necessary equations and solve them to get explicit solutions of the oscillator’s trajectory, velocity, wave function, etc. in closed-form, analytic functions, even without the use of computers for numerical methods. Thus, this creates the necessary motivation to study this fascinating and deceptively simple system. Let’s now add damping into the system, as virtually all objects undergo damping in real life due to various dissipation forces such as air resistance and heat loss.¹

In classical mechanics, the phenomenological approach to resolve the damping is as follows: first, one takes into account the damping of the oscillator by adding a “friction-like” term $-\eta\dot{x}$, where η is the drag coefficient and x is the position of the oscillator (so $\dot{x} \equiv dx/dt$ is the oscillator’s velocity). This assumes that the

¹An exotic example of an object which undergoes no damping is a superconductor; nevertheless we will not consider it in this thesis.

damping is linear, but the damped harmonic oscillator is still exactly solvable if it were quadratic as well ($-\eta\dot{x}^2$) [1]. In quantum mechanics however, the addition of damping is nontrivial, as it is a nonconservative force (which means that there is no potential function which gives rise to this force, and one cannot just immediately write down the Schrödinger equation and proceed with the usual methods of attack). This in turn makes models of damped quantum oscillators hard to develop, and in fact, no model so far addresses all of its issues (namely, broken time-reversal symmetry, due to the addition of this damping term). In this thesis, I will reformulate a model, namely the Lamb model [3], by using a Lagrangian framework and address the coupling between the bead and the string in the form of a holonomic constraint, and study it classically, semiclassically, and quantum mechanically (where quantization of the model is attempted). In the classical approach, I extract the equations of motion and solve it exactly with the use of integral transform methods. Then, I study the problem semiclassically, by using Legendre transformations to convert the individual Lagrangians to their corresponding Hamiltonians. The use of perturbation theory is applied to extract the first-order transition rate of the bead from its ground state to its first excited state, following the method analogous to calculating emission and absorption of light [2]. Then, in the quantum mechanical treatment, I continue using the previously-calculated Hamiltonians and perturbation theory to recalculate the first-order transition rate of the bead, but this time using Fermi's golden rule, to reaffirm the previous results.

1.1 HISTORY

Any system that's displaced from its stable equilibrium position will be subject to oscillations. If this displacement is small enough, the oscillations will be "simple harmonic" [1]. The differential equation corresponding to simple harmonic motion is

of the form

$$m \frac{d^2 x}{dt^2} = -kx. \quad (1.1)$$

In this type of motion, the potential energy varies quadratically with the distance, i.e.,

$$U(x) = \frac{kx^2}{2}, \quad (1.2)$$

where k is the effective force constant in the problem (for a mass–spring system, k would be the spring constant). We can also write down the corresponding force law:

$$F(x) = -kx. \quad (1.3)$$

While people have observed oscillations for centuries, the earliest known work for which the simple harmonic oscillator problem was “solved” was in 1673 by Christiaan Huygens [4]. While Huygens did not write down Eq. (1.1), or even use Newton’s laws, he still managed to study the motion of pendulums, and was able to use the conservation of mechanical energy to derive (what would be equivalent to today) the oscillations for a simple pendulum of length ℓ :

$$T = 2\pi \sqrt{\frac{\ell}{g}}. \quad (1.4)$$

Further developments on the problem were during the 18th century, with the independent efforts of Euler [5] and Bernoulli [6]. In fact, Euler’s work dealt with more general differential equations occurred in mechanical systems, and he managed to obtain solutions to Eq. (1.1) and the forced harmonic oscillator equation

$$m \frac{d^2 x}{dt^2} + kx = F \sin(\omega t) \quad (1.5)$$

by quadrature, and reaffirmed the phenomenon of resonance [5].

In real life, no oscillations are fully undamped.² The very fact that air resistance and energy transfer exists means that there will always be resistive forces in a system that undergoes oscillations.³ These resistive forces damp the oscillations by energy loss, which means that over time, the oscillations become weaker and their amplitude decreases. As such, Eq. (1.1) is mostly a theoretical tool to study cases in the limit where damping goes to zero. Nevertheless, in some cases, damping is an important factor, and with the help of the theory of power series and differential equations, it can be studied in simple cases. We present the derivation analogous to the one in Taylor [1].

Using Taylor series, any function of velocity can be expanded in the following form:

$$f(v) = a + bv + cv^2 + \dots, \quad (1.6)$$

where a , b , and c are constants to be determined. For low enough velocity, we take $a = 0$ (as the damping force is zero when the object is stationary), and $c = 0$ (as we can neglect the quadratic term). This allows us to model the damping force as

$$f(v) = bv. \quad (1.7)$$

As the damping force always points opposite to the object's direction of motion, we need b to be negative, so let $b = -\eta$, where η is some positive drag coefficient. Using Newton's second law, Eq. (1.1) turns into

$$m \frac{d^2x}{dt^2} = -\eta \frac{dx}{dt} - kx. \quad (1.8)$$

²It has been theorized that time crystals may exhibit undamped oscillations; nevertheless, for our purposes, we will not consider them here.

³Sufficiently rigid systems may undergo no oscillations at all, but we are not talking about them here.

The theory of differential equations tells us that this equation has two linearly independent solutions, whose behaviour depends on the strength of η , our damping parameter. To be more precise, there are three cases: (i) light damping, where $\eta < 2m\omega_0$ (ω_0 is the undamped frequency of oscillations), (ii) critical damping, where $\eta = 2m\omega_0$, and (iii) strong damping, where $\eta > 2m\omega_0$. We'll examine only the first case (light damping) for the classical Lamb model.

By the early 20th century, the development of quantum mechanics spearheaded by Born, Planck, Bohr, Schrödinger, Dirac, Heisenberg, Einstein, and others (in no particular order) lead to the study of the behaviour of atoms, molecules, and waves at the microscopic level. With that came one of the most important equations of modern physics, the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \right] \Psi(x, t). \quad (1.9)$$

One gets $\Psi(x, t)$ by solving Eq. (1.6), which is called the **wave function** of the system, and that allows one to find out everything about the system's quantum state. In the case where the potential is independent of time, one can solve for the stationary state $\psi(x)$, which is obtained by solving the time-*independent* Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi. \quad (1.10)$$

This can be recast in the form of an eigenvalue equation:

$$\hat{H}\psi = E\psi, \quad (1.11)$$

where

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \quad (1.12)$$

is the Hamiltonian operator.

For the simple harmonic oscillator potential, where $V(x) = kx^2/2$, the time-independent Schrödinger equation has been solved exactly; an early historical presentation is presented in [7] and [8], while a more accessible, modern treatment is showcased in [2].

Notice that all information about the potential is contained in $V(x)$. Therein lies the problem of the addition of damping into the quantum formulation of the problem. When we introduced damping classically, notice how it was not derived from a conservative force, i.e., there was no “potential energy function” from which we could differentiate it from. Note that *does not* mean that we must immediately approach the damped harmonic oscillator problem in another way. For example, Goldstein, Poole, and Safko [9] consider the treatment of *classical* dissipative systems in the Hamiltonian formulation, and the effective potential can be calculated, and in some cases may even be complex-valued (non-Hermitian). The *quantum* theory of such systems has been spearheaded in recent decades by Carl Bender [10].

Two noteworthy alternative approaches to this problem were done in the 20th century as follows. The first was to add damping in the form of energy exchange between two harmonic oscillators, and then treat the whole system quantum mechanically. This has been done by Leggett et al. [11] in 1987, where he considers a bath of oscillators exchanging energy. Another approach, pioneered by Feshbach [12] is to consider a system with complementary dynamics that of the damped harmonic oscillator. However, both models assume an interaction term in the Hamiltonian, which is contrary to the approach shown in this thesis.

In this thesis, I will outline a way to study this problem quantum mechanically, which involves reformulating the Lamb model, where the interaction arises from enforcing a holonomic constraint between the bead and the string.

1.2 SIGNIFICANCE

While this may seem like a very rich and interesting problem which is only of theoretical importance, it turns out that this problem has applications in today's world. With the rapid development of quantum technology, some quantum devices (such as the ones used for quantum sensing), are made with atoms bound to membranes. The very fact that the atoms are bound to membranes means that an excitation caused by infrared radiation (for example) would make them oscillate, while the membrane would dampen their oscillations as a result of the atom–membrane coupling. As these membranes and atoms are on the microscopic scale, a quantum mechanical treatment must be done on the system to understand its dynamics correctly. This reinforces the need of a full quantum mechanical treatment of the damped harmonic oscillator problem; an added bonus is the fact that the system previously mentioned has damping in the form analogous to the one in the Lamb model.

CHAPTER 2

THE LAMB MODEL

I plan to use the Lamb model to study the damped quantum harmonic oscillator. A diagram of this model is shown in Fig. 2.1.

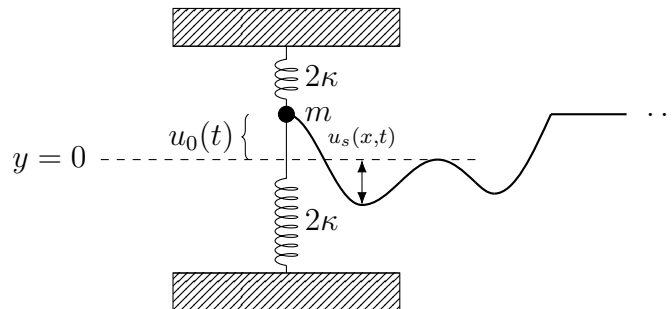


Figure 2.1: Setup of the Lamb model. A bead with mass m can vibrate in the vertical direction, and a long string under tension is directly attached to the bead.

This setup was first thought of by Horace Lamb in 1900 [3], where a string of semi-infinite length is attached to a simple harmonic oscillator. In this setup, the motion of the bead drives transverse waves on the string and the string itself damps oscillations of the bead. Lamb’s original approach to this problem involves solving the classical equations of motion for the “open” system, where there are only outgoing elastic waves on the string. Here, I am interested in developing an approach from “first

principles” which *does not* involve adding the “friction-like” term mentioned previously, but formulate the Lagrangian for the system subject to a holonomic constraint, namely the string is physically attached to the bead. This way, the interaction term follows naturally from the coupling. With that in mind, let’s now perform a thorough analysis of this model, in three regimes of interest: first, the purely classical case, where the string and bead are both treated only using classical mechanics; second, the semiclassical case, where the string is treated classically but the bead quantum mechanically; and third, the purely quantum mechanical case, where both the string and bead are treated only using quantum mechanics.

The reason for the choosing of this model to study the damped harmonic oscillator problem is due to its simple design while still exhibiting the richness of oscillation theory. Combined with the fact that quantization of the bead alone is possible [2], the system’s damping arises as a constraint in its dynamics rather than employing null Lagrangians, fractional calculus, or other esoteric techniques, analysis of this model can be fully done with common topics covered in classical mechanics, quantum mechanics, and field theory. Furthermore, as mentioned previously, this model is a one-dimensional, simplified version of real-life quantum sensing devices.

CHAPTER 3

THE CLASSICAL TREATMENT

In classical mechanics, the dynamics of the model is best treated with the Lagrangian formulation. The total Lagrangian of the system is given by

$$L = L_b + L_s. \quad (3.1)$$

The Lagrangian of the bead is simple:

$$L_b = \frac{m\dot{u}_0^2}{2} - \frac{\kappa u_0^2}{2}. \quad (3.2)$$

Meanwhile, the Lagrangian of the string, which has length ℓ , uniform lineal mass density σ , and is under tension τ , is given by

$$L_s = \int_0^\ell \mathcal{L}_s dx = \int_0^\ell \left[\frac{\sigma}{2} \left(\frac{\partial u_s}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial u_s}{\partial x} \right)^2 \right] dx \quad (3.3)$$

where $u(x, t)$ is the string's vertical displacement in terms of its horizontal position x and time t relative to the line $y = 0$, and \mathcal{L}_s is the string's **Lagrange density**.

A third term arises by enforcing the constraint that the string is physically attached to the bead. This means that $u_0(t) = u_s(0, t)$ for all time. In equation form,

we can therefore introduce a **Lagrange multiplier** λ in our equations of motion as follows:

$$L_{\text{int}} = \lambda[u_0(t) - u_s(0, t)]. \quad (3.4)$$

Since $u_0(t)$ models the bead's position and $u_s(x, t)$ models the string's position, we can add each individual piece to the bead and string's Lagrangians separately, so that with the constraint, the new Lagrangians are

$$\begin{aligned} L_b &= \frac{m\dot{u}_0^2}{2} - \frac{\kappa u_0^2}{2} + \lambda u_0 \\ L_s &= \int_0^\ell \left[\frac{\sigma}{2} \left(\frac{\partial u_s}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial u_s}{\partial x} \right)^2 - \lambda \delta(x) u_s \right] dx. \end{aligned} \quad (3.5)$$

Now, Hamilton's principle states that the dynamics of this model (and its subsequent equations of motion) are found by extremizing the action between two time points (call them t_1 and t_2):

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (3.6)$$

with $\delta u(x, t_1) = \delta u(x, t_2) = \delta u_0(x, t_1) = \delta u_0(x, t_2) = 0$.

We therefore find that

$$\delta \int_{t_1}^{t_2} L_b dt + \delta \int_{t_1}^{t_2} L_s dt = 0. \quad (3.7)$$

Applying the definition of the variation, we get that

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial L_b}{\partial \dot{u}_0} \right) \delta \dot{u}_0 + \left(\frac{\partial L_b}{\partial u_0} \right) \delta u_0 \right] dt + \int_0^\ell \int_{t_1}^{t_2} \left[\left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) \delta \dot{u}_s + \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) \delta u'_s \right] dx dt = 0. \quad (3.8)$$

Now, we have to use integration by parts carefully on the following terms:

$$\begin{aligned}
\int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial \dot{u}_0} \right) \delta \dot{u}_0 dt &= \left(\frac{\partial L_b}{\partial \dot{u}_0} \right) \delta u_0 \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L_b}{\partial \dot{u}_0} \right) \delta u_0 dt \\
\int_0^\ell \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) \delta \dot{u}_s dx dt &= \int_0^\ell \left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) \delta u_s dx \Big|_{t_1}^{t_2} - \int_0^\ell \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) \delta u_s dx dt \quad (3.9) \\
\int_0^\ell \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) \delta u'_s dx dt &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) \delta u_s dt \Big|_0^\ell - \int_0^\ell \int_{t_1}^{t_2} \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) \delta u_s dx dt.
\end{aligned}$$

All the first (boundary) terms disappear either due to variation endpoint or boundary conditions of the system, so we are just left with the second terms (the integrals). Plugging these newfound relations carefully into Eq. (3.8) and combining like-terms in δu and δu_0 , we get that

$$\int_{t_1}^{t_2} \left[\frac{\partial L_b}{\partial u_0} - \frac{d}{dt} \left(\frac{\partial L_b}{\partial \dot{u}_0} \right) \right] \delta u_0 dt - \int_0^\ell \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) \right] \delta u_s dx dt = 0. \quad (3.10)$$

Since δu and δu_0 are *independent* variations, the only way for Eq. (3.10) to be satisfied is if the terms in both square brackets are zero, and so we get our Euler–Lagrange equations for this system:

$$\begin{aligned}
\frac{\partial L_b}{\partial u_0} - \frac{d}{dt} \left(\frac{\partial L_b}{\partial \dot{u}_0} \right) &= 0 \\
\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}_s}{\partial \dot{u}_s} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}_s}{\partial u'_s} \right) &= 0.
\end{aligned} \quad (3.11)$$

If we now evaluate these equations of motion by going back to the individual Lagrangians, we will find that

$$\begin{aligned}
m\ddot{u}_0 + \kappa u_0 &= \lambda \\
\sigma \frac{\partial^2 u_s}{\partial t^2} - \tau \frac{\partial^2 u_s}{\partial x^2} &= -\lambda \delta(x).
\end{aligned} \quad (3.12)$$

3.1 THE CLASSICAL EQUATIONS OF MOTION

To solve these coupled differential equations, it is best to use the method of Laplace transforms to efficiently incorporate the initial conditions. This will uncouple the equations by transforming them into a system of *algebraic* ones. But, first let's find λ . This can be found by integrating the second equation in Eq. (3.12) from $-\epsilon$ to $+\epsilon$ in x and then taking the limit $\epsilon \rightarrow 0$. One then finds that

$$\lambda = \tau \frac{\partial u_s}{\partial x}(0, t). \quad (3.13)$$

This means that the equations of motion become

$$\begin{aligned} \ddot{u}_0 + \omega_0^2 u_0 &= \frac{\tau}{m} \frac{\partial u_s}{\partial x}(0, t) \\ \frac{1}{c^2} \frac{\partial^2 u_s}{\partial t^2} - \frac{\partial^2 u_s}{\partial x^2} &= -\delta(x) \frac{\partial u_s}{\partial x}(0, t), \end{aligned} \quad (3.14)$$

where we have defined $\omega_0 \equiv \sqrt{\kappa/m}$ to be the undamped frequency of oscillations and $c = \sqrt{\tau/\sigma}$ to be the **wave speed**.

The (unilateral) Laplace transform of a function $f(t)$ is defined by

$$\bar{f}(s) \equiv \int_0^\infty f(t) e^{-st} dt, \quad (3.15)$$

which converts the function from the time domain to the (complex) frequency domain. This can further be extended to fields, and in this case, we can define the Laplace transform of our displacement field $u_s(x, t)$ as follows:

$$\bar{u}_s(x, s) \equiv \int_0^\infty u_s(x, t) e^{-st} dt. \quad (3.16)$$

There are a couple of nice identities with Laplace transforms; below are some common

ones that we'll use when deriving the equations of motion:

$$\bar{u}_s(x, s) = s\bar{u}_s(x, s) - u_s(x, 0) \quad (3.17)$$

$$\bar{\dot{u}}_s(x, s) = s^2\bar{u}_s(x, s) - su_s(x, 0) - \dot{u}_s(x, 0) \quad (3.18)$$

If one has information about the Laplace-transformed field, say $\bar{u}_s(x, s)$, then one can invert the transform to find $u_s(x, t)$ by either using a computer program (such as **Mathematica**) or performing the Bromwich integral:

$$u_s(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{u}_s(x, s) e^{st} ds, \quad (3.19)$$

where the contour Γ consists of the vertical line γ in the complex s -plane such that it lies on the right of **all** the singularities of $\bar{u}_s(x, s)$.

Armed with these tools in mind, let's now solve Eq. (3.14). The two equations transform into the following ones:

$$\begin{aligned} s^2\bar{u}_0(s) - as + \omega_0^2\bar{u}_0(s) &= \frac{\tau}{m} \frac{\partial \bar{u}_s}{\partial x}(0, s) \\ \frac{1}{c^2} \left(s^2\bar{u}_s(x, s) - as \right) - \frac{\partial^2 \bar{u}_s}{\partial x^2} &= -\delta(x) \frac{\partial \bar{u}_s}{\partial x}(0, s) \end{aligned} \quad (3.20)$$

because we have chosen the following initial conditions: $u_0(0) = u_s(x, 0) = a$ and $\dot{u}_0(0) = \dot{u}_s(x, 0) = 0$. These conditions correspond to giving the bead an initial energy and having the bead radiate that energy away by the transverse waves on the string.

Rearranging the second of Eq. (3.20), we get that

$$\frac{\partial^2 \bar{u}_s}{\partial x^2} - \left(\frac{s}{c} \right)^2 \bar{u}_s(x, s) - \delta(x) \frac{\partial \bar{u}_s}{\partial x}(0, s) = -\frac{as}{c^2}. \quad (3.21)$$

To solve this inhomogeneous partial differential equation for $x > 0$, let $\bar{u}_s(x, s) =$

$\bar{u}_h(x, s) + \bar{u}_p(x, s)$ where $\bar{u}_h(x, s)$ satisfies

$$\frac{\partial^2 \bar{u}_h}{\partial x^2} - \left(\frac{s}{c}\right)^2 \bar{u}_h(x, s) = 0 \quad (3.22)$$

and $\bar{u}_p(x, s)$ satisfies

$$\frac{\partial^2 \bar{u}_p}{\partial x^2} - \left(\frac{s}{c}\right)^2 \bar{u}_p(x, s) = -\frac{as}{c^2}. \quad (3.23)$$

Eq. (3.22) has solutions of the form

$$\bar{u}_h(x, s) = A(s) \sinh\left(\frac{sx}{c}\right) + B(s) \cosh\left(\frac{sx}{c}\right). \quad (3.24)$$

Meanwhile, Eq. (3.24) has a particular solution of the form

$$\bar{u}_p(x, s) = \frac{a}{s}. \quad (3.25)$$

We therefore have the general solution for the Laplace-transformed string displacement field $\bar{u}_s(x, s)$:

$$\bar{u}_s(x, s) = A(s) \sinh\left(\frac{sx}{c}\right) + B(s) \cosh\left(\frac{sx}{c}\right) + \frac{a}{s}. \quad (3.26)$$

Therefore, the first of Eq. (3.20) becomes

$$(s^2 + \omega_0^2) \bar{u}_0(s) = as + A(s) \cdot \frac{\tau s}{mc}, \quad (3.27)$$

leading to

$$\bar{u}_0(s) = \frac{as}{s^2 + \omega_0^2} + A(s) \cdot \frac{\tau s}{mc(s^2 + \omega_0^2)}. \quad (3.28)$$

From the constraint, $\bar{u}_0(s) = \bar{u}_s(0, t)$, so we require that

$$\bar{u}_s(0, s) = \frac{as}{s^2 + \omega_0^2} + A(s) \cdot \frac{\tau s}{mc(s^2 + \omega_0^2)} = B(s) + \frac{a}{s}. \quad (3.29)$$

In the original Lamb model, the string is of infinite length, so if we take the limit as $\ell \rightarrow \infty$, only outgoing waves will be on the string, and we can use the **Sommerfeld radiation condition** [12] to find that

$$\frac{\partial u_s}{\partial x}(0, t) = -\frac{1}{c} \frac{\partial u_s}{\partial t}(0, t); \quad (3.30)$$

hence

$$\frac{\partial \bar{u}_s}{\partial x}(0, s) = -\frac{1}{c} [s\bar{u}_s(0, s) - a]. \quad (3.31)$$

From our expressions of $\bar{u}_s(x, s)$ and $\bar{u}_0(s)$, we find that

$$A(s) \cdot \frac{s}{c} = -\frac{1}{c} \left(s \left[B(s) + \frac{a}{s} \right] - a \right), \quad (3.32)$$

which gives us that $A(s) = -B(s)$. This means that

$$\frac{as}{s^2 + \omega_0^2} + A(s) \cdot \frac{\tau s}{mc(s^2 + \omega_0^2)} = \frac{a}{s} - A(s), \quad (3.33)$$

or

$$A(s) = \frac{a\omega_0^2}{s(s^2 + 2\nu s + \omega_0^2)}, \quad (3.34)$$

where we have defined the damping rate to be $\nu \equiv \tau/(2mc)$.

Substitution of this expression of $A(s)$ into Eq. (3.28) yields

$$\bar{u}_0(s) = \frac{a(s^3 + 2\nu s^2 + \omega_0^2 s + 2\nu\omega_0^2)}{(s^2 + \omega_0^2)(s^2 + 2\nu s + \omega_0^2)}, \quad (3.35)$$

with

$$\bar{u}_s(x, s) = \frac{a\omega_0^2}{s(s^2 + 2\nu s + \omega_0^2)} \left[\sinh\left(\frac{sx}{c}\right) - \cosh\left(\frac{sx}{c}\right) \right] + \frac{a}{s}. \quad (3.36)$$

The displacement field is therefore

$$u_s(x, t) = a \left[1 + \theta(t - x/c) \left(\frac{\omega_0^2 e^{-\nu(t-x/c)}}{\sqrt{(\omega_0^2 + \nu^2)^2 + \nu^2 \omega^2}} \cos[\omega(t - x/c) - \varphi] - 1 \right) \right], \quad (3.37)$$

where $\omega = \sqrt{\omega_0^2 - \nu^2}$, $\varphi = \tan^{-1}[\nu\omega/(\omega_0^2 + \nu^2)]$, and $\theta(x)$ is the **Heaviside unit step function**.

In the light damping or **underdamped** regime, we have that $\nu^2 < \omega_0^2$, so we find that the poles become

$$s_{\pm} = -\nu \pm i\sqrt{\omega_0^2 - \nu^2} = -\nu \pm i\omega. \quad (3.38)$$

Using the contour shown in Fig 3.2 to perform the contour integration, we get that

$$u_0(t) = ae^{-\nu t} \left[\frac{\nu}{\omega} \sin(\omega t) + \cos(\omega t) \right]. \quad (3.39)$$

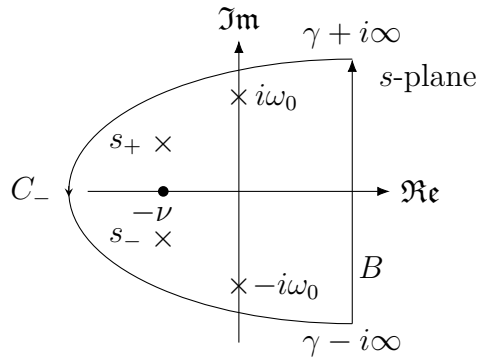


Figure 3.2: Closed contour used to invert Eq. (3.35) to get $u_0(t)$ for light damping. B is the Bromwich line that ensures the regularity of the integral, while C_- is an additional semicircle of infinite radius used to close the contour; Jordan's lemma guarantees that it adds no contribution to the overall integral for $t > 0$.

This is exactly the equation of damped harmonic motion [1]. A graph of the position function of $u_0(t)$ is shown in Fig 3.3.

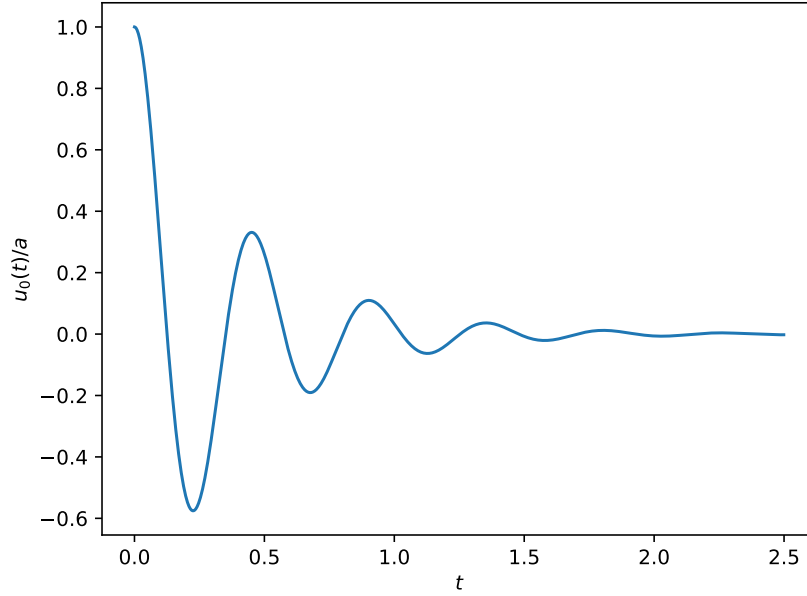


Figure 3.3: Graph of the bead's position $u_0(t)$ in units of a against time t for light damping, with the following parameters: $\kappa = 50 \text{ N/m}$, $\tau = 0.5 \text{ N}$, $m = 0.25 \text{ kg}$, $\sigma = 3 \text{ kg/m}$, and $0 \leq t \leq 2.5 \text{ s}$. These values were chosen so that $\nu < \omega_0$.

The displacement field for this case is given by

$$u_s(x, t) \approx \begin{cases} ae^{-\nu(t-x/c)} \cos[\omega_0(t - x/c)], & x < ct, \\ a, & x > ct. \end{cases} \quad (3.40)$$

A plot of the field is shown in Fig 3.4, at a chosen time $t = 1$ s.

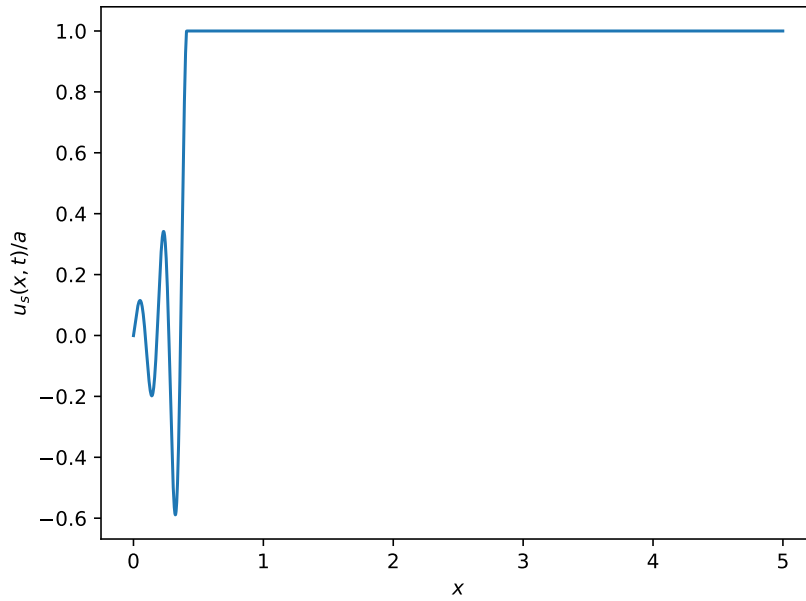


Figure 3.4: Graph of the field $u_s(x, t)$ in units of a against position x for light damping, with the following parameters: $\kappa = 50$ N/m, $\tau = 0.5$ N, $m = 0.25$ kg, $\sigma = 3$ kg/m, $t = 1$ s, and $0 \leq x \leq 5$ m. These values were chosen so that $\nu < \omega_0$.

(The reason for the discontinuity in the slope of $u_s(x, t)$ is because news of the bead's oscillations have not reached parts of the string beyond $x = ct$.)

3.2 TIME DEPENDENCE OF THE ENERGY OF THE BEAD

As the bead oscillates back and forth before its oscillations die out as a result of the damping force from the string's back action, its energy is therefore not constant, as

it would be for an undamped simple harmonic oscillator [1]. Therefore, it is of great interest to calculate the total energy of the bead, as it will give us information about how it loses energy over time.

From classical mechanics, the total energy of the bead (as a function of time) is given by

$$E(t) = \frac{m\dot{u}_0^2}{2} + \frac{\kappa u_0^2}{2}. \quad (3.41)$$

We have found $u_0(t)$ in the previous section, but for this calculation, it'll be more prudent to rewrite it in the following way:

$$u_0(t) = U(t) \cos(\omega t - \phi), \quad (3.42)$$

where

$$\begin{aligned} U(t) &= \frac{a\omega_0}{\omega} e^{-\nu t}, \\ \phi &= \tan^{-1} \left(\frac{\nu}{\omega} \right). \end{aligned} \quad (3.43)$$

For proof of the equivalence between this and Eq. (3.39), see the Appendix. The first derivative of $\dot{u}_0(t)$ is given by

$$\dot{u}_0(t) = -U(t)[\nu \cos(\omega t - \phi) + \omega \sin(\omega t - \phi)]. \quad (3.44)$$

Again, the same trick that converted $u_0(t)$ into a single sine/cosine function can be used in this case, and we find that

$$\dot{u}_0(t) = -\omega_0 U(t) \sin(\omega t). \quad (3.45)$$

We therefore find that the time-dependent energy of the bead is given by

$$E(t) = \frac{\kappa}{2} \left(\frac{a\omega_0}{\omega} \right)^2 e^{-2\nu t} [\sin^2(\omega t) + \cos^2(\omega t - \phi)]. \quad (3.46)$$

A graph of the energy is shown in Fig 3.5.

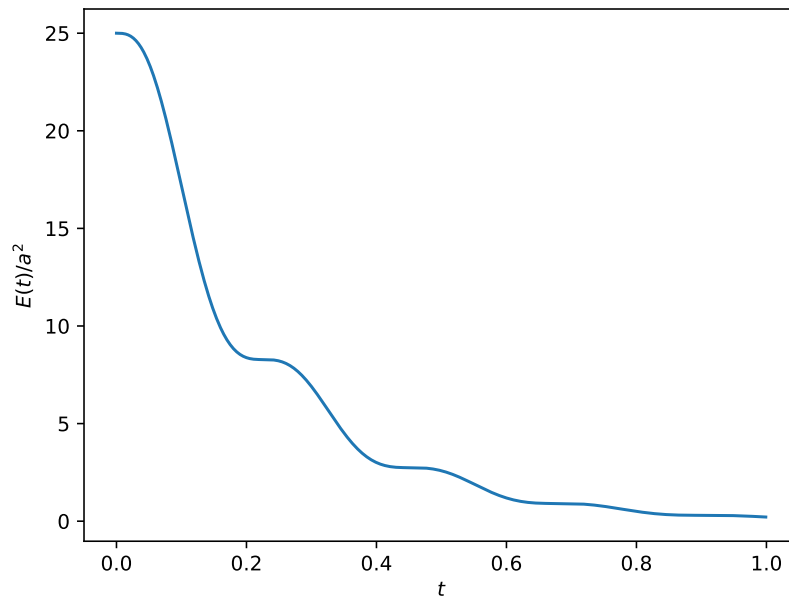


Figure 3.5: Energy of the bead $E(t)$ in units of a^2 against time t for light damping, with the following parameters: $\kappa = 50 \text{ m}$, $\tau = 0.5 \text{ N}$, $m = 0.25 \text{ kg}$, $\sigma = 3 \text{ kg/m}$, and $0 \leq t \leq 1 \text{ s}$. These values were chosen so that $\nu < \omega_0$.

3.3 ACOUSTIC POWER RADIATED

After calculating the energy of the bead, one can then take the first time derivative to find the power radiated by the bead as a function of time, and then calculate its

time average over an oscillation. Again, the steps below show the working out for the weakly-damped bead, but the same procedure can be applied to the other two damping regimes.

We know that

$$P(t) = \dot{E}(t) = \dot{u}_0(m\ddot{u}_0 + \kappa u_0). \quad (3.47)$$

If this were an undamped oscillator, then $P(t) = 0$, as for an undamped harmonic oscillator, the equation of motion is $m\ddot{u}_0 = -\kappa u_0$.

The second derivative of \ddot{u}_0 is a bit more complicated, but applying the trick previously mentioned again gives us that

$$\ddot{u}_0(t) = -\omega_0^2 U(t) \cos(\omega t + \phi). \quad (3.48)$$

Combining this with $\dot{u}_0(t)$ and $u_0(t)$ found earlier, and after simplifying the algebra, one will find that

$$P(t) = -2\kappa \left(\frac{a\omega_0}{\omega} \right)^2 \nu e^{-2\nu t} \sin^2(\omega t). \quad (3.49)$$

This gives the power that the bead loses; to find the power that the bead radiates over time, we take the absolute value of Eq. (3.49) to find that

$$P_{\text{rad}}(t) = 2\kappa \left(\frac{a\omega_0}{\omega} \right)^2 \nu e^{-2\nu t} \sin^2(\omega t). \quad (3.50)$$

Its graph is shown in Fig 3.6.

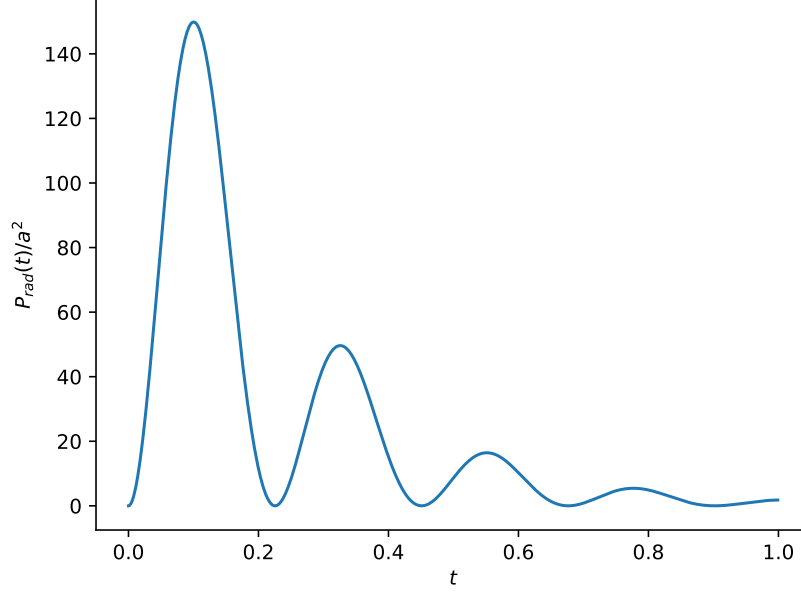


Figure 3.6: Power radiated of the bead $P_{\text{rad}}(t)$ in units of a^2 against time t , with the following parameters: $\kappa = 50$ N, $\tau = 0.5$ N/m, $m = 0.25$ kg, $\sigma = 3$ kg/m, and $0 \leq t \leq 1$ s. These values were chosen so that $\nu < \omega_0$.

Let's now calculate its time average over an oscillation. The time-averaged power is given by

$$\langle P_{\text{rad}} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} P_{\text{rad}}(t) dt. \quad (3.51)$$

In this case, we find that

$$\langle P_{\text{rad}} \rangle = \frac{\kappa\nu(a\omega_0)^2}{\pi\omega} \int_0^{2\pi/\omega} e^{-2\nu t} \sin^2(\omega t) dt. \quad (3.52)$$

This integral can be done analytically, and I'll simply quote the result:

$$\int_0^{2\pi/\omega} e^{-2\nu t} \sin^2(\omega t) dt = \frac{1}{4\nu} \left(\frac{\omega}{\omega_0}\right)^2 (1 - e^{-4\pi\nu/\omega}). \quad (3.53)$$

Therefore, the exact time-averaged power radiated over an oscillation is

$$\langle P_{\text{rad}} \rangle = \frac{\kappa a^2 \omega}{4\pi} (1 - e^{-4\pi\nu/\omega}). \quad (3.54)$$

Its graph is shown in Fig 3.7.

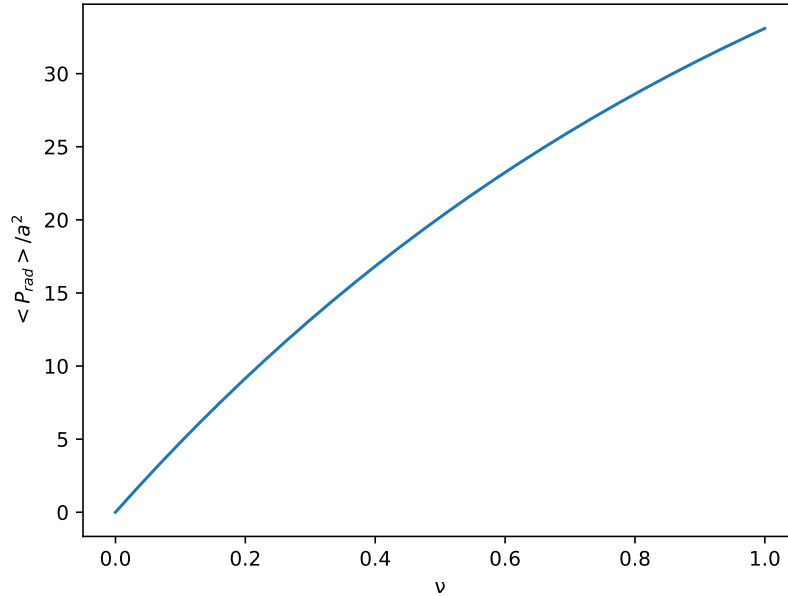


Figure 3.7: Time-averaged power radiated $\langle P_{\text{rad}} \rangle$ in units of a^2 against the damping rate ν , with the following parameters: $\kappa = 50$ N/m, $m = 0.25$ kg, $\sigma = 3$ kg/m, and $0 \leq \nu \leq 1$ s $^{-1}$. Note that the range of ν is such that it is always less than ω_0 , as we derived these results from the form of $u_0(t)$ for light damping.

For very light damping (where $\nu \ll \omega_0$), we can expand the exponential factor using Taylor series to its first two terms, and we find that

$$\langle P_{\text{rad}} \rangle \approx \kappa a^2 \nu = E_{\text{initial}} \cdot 2\nu, \quad (3.55)$$

where $E_{\text{initial}} = \kappa a^2/2$ is the bead's total initial energy stored as it was displaced from its equilibrium position by a distance of a vertically. This has a very nice interpretation; it states that for very light damping, the average power radiated by the bead over an oscillation is simply twice the product of the initial energy of the bead and the damping rate.

CHAPTER 4

THE SEMICLASSICAL TREATMENT

In the semiclassical treatment of the problem, we pick one of the components in the whole system to be treated classically, and the other quantum mechanically. I have chosen to treat the bead quantum mechanically, because decoupling the bead from the string gives us an ordinary one-dimensional quantum harmonic oscillator.

4.1 QUANTUM LAMB MODEL

We can obtain the Hamiltonian of the system from the Lagrangian in the canonical fashion. The total Hamiltonian has the form

$$H = H_b + H_s + H_{\text{int}} \tag{4.1}$$

where H_b is the bead's Hamiltonian, H_s is the string's Hamiltonian, and H_{int} is the interaction Hamiltonian. The Hamiltonians can be found by applying Legendre transformations on the system Lagrangian (which is a function of the generalized

coordinates $\{q_i\}$ and their corresponding time derivatives $\{\dot{q}_i\}$, defined by

$$H \equiv \sum_i p_i \frac{\partial q_i}{\partial t} - L. \quad (4.2)$$

In this case, we get that

$$\begin{aligned} H_b &= \frac{m\dot{u}_0^2}{2} + \frac{\kappa u_0^2}{2} \\ H_s &= \int_0^\ell \left[\frac{\sigma}{2} \left(\frac{\partial u_s}{\partial t} \right)^2 + \frac{\tau}{2} \left(\frac{\partial u_s}{\partial x} \right)^2 \right] dx \\ H_{\text{int}} &= -\tau \frac{\partial u_s}{\partial x}(0, t) u_0. \end{aligned} \quad (4.3)$$

We can keep the $\partial u_s / \partial x$ term as it is (because we are treating it classically), and then rewrite the bead's vertical displacement $u_0(t)$ in terms of **creation** and **annihilation operators** a and a^\dagger [2, 14]:

$$u_0 = \sqrt{\frac{\hbar}{2m\omega_0}} (a + a^\dagger) \quad (4.4)$$

where

$$\begin{aligned} a &= \frac{1}{\sqrt{2\hbar m\omega_0}} (ip + m\omega_0 u_0) \\ a^\dagger &= \frac{1}{\sqrt{2\hbar m\omega_0}} (-ip + m\omega_0 u_0), \end{aligned} \quad (4.5)$$

and

$$[a, a^\dagger] = 1. \quad (4.6)$$

Now, consider an incident standing wave coming off from $x = \infty$ towards the bead (which is located at $x = 0$), i.e., take

$$u_s(x, t) = U \sin(kx) \cos(\omega t) \quad (4.7)$$

(for some amplitude U), so that

$$\frac{\partial u_s}{\partial x}(0, t) = kU \cos(\omega t). \quad (4.8)$$

The interaction Hamiltonian is therefore

$$H_{\text{int}} = -kU\tau \sqrt{\frac{\hbar}{2m\omega_0}} (a + a^\dagger) \cos(\omega t) = V \cos(\omega t). \quad (4.9)$$

The effects of the interaction term can now be analyzed using **time-dependent perturbation theory** for a harmonic interaction [2].

4.2 TRANSITION RATES

Following the approach of Griffiths and Schroeter [2], we can use time-dependent perturbation theory to calculate the absorption rate for the bead, which initially, say, starts out in its ground state, $|0\rangle$, and would like to transit into its first excited state, $|1\rangle$, as a result of the perturbation. This would be common in a system where an atom is sitting at a low temperature (so that it is in its ground state; this commonly occurs in Bose–Einstein condensates), and becomes excited by some external source, say a laser for example. This means that at $t = 0$, the (time-dependent) coefficients $c_0(t)$ and $c_1(t)$ take on the following values:⁴

$$\begin{aligned} c_0(0) &= 1 \\ c_1(0) &= 0. \end{aligned} \quad (4.10)$$

⁴The relevant selection rule for this transition would be $\Delta n = 1$, where n labels the states of the bead. (The n^{th} state of the bead is thus written as $|n\rangle$.) Since the bead starts in the ground state, it can only transit to its first excited state in this calculation.

However, we know that $c_1(t)$ is equal to (in first order) [2]:

$$c_1(t) = c_1^{(1)}(t) = -\frac{iV}{\hbar} \int_0^t \langle 1 | \cos(\omega t') | 0 \rangle e^{i\omega_{10}t'} dt'$$

where

$$\omega_{10} \equiv \frac{E_1 - E_0}{\hbar} = \omega_0 \quad (4.11)$$

is the **Bohr frequency** of the atom in units of \hbar . (Recall that the energy spectrum of the quantum harmonic oscillator is $E_n = (n + 1/2)\hbar\omega$ for $n \in \mathbb{N}_0$.)

We now find that

$$c_1(t) \approx \frac{ikU\tau}{\sqrt{2m\hbar\omega_0}} \langle 1 | (a + a^\dagger) | 0 \rangle \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt'. \quad (4.12)$$

But,

$$\langle 1 | (a + a^\dagger) | 0 \rangle = \langle 1 | 1 \rangle = 1 \quad (4.13)$$

due to the orthonormality of the eigenstates, so we have that

$$c_1(t) \approx \frac{ikU\tau}{\sqrt{2m\hbar\omega_0}} \cdot \frac{\sin[(\omega_0 - \omega)t/2]}{\omega_0 - \omega} e^{i(\omega_0 - \omega)t/2} \quad (4.14)$$

for $\omega \approx \omega_0$ (this is the resonance condition). Squaring its modulus will give us the transition probability for the bead to go from its ground state to the first excited state:

$$P_{0 \rightarrow 1}(t) = |c_1(t)|^2 \approx \frac{(kU\tau)^2}{2m\hbar\omega_0} \cdot \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}. \quad (4.15)$$

(The reason we can make the assumption $\omega \approx \omega_0$ is because we are interested in absorption, which has the greatest effect in the resonance condition.)

Now, the square of the amplitude U is related to the energy density of vibrations \mathcal{U} . Let \mathcal{T} denote the kinetic energy density; it's given by the first term in square

brackets of Eq. (3.5). From the virial theorem [1, 2],

$$\langle \mathcal{T} \rangle = \langle \mathcal{U} \rangle, \quad (4.16)$$

so

$$U^2 = \frac{4}{\sigma\omega^2} \langle \mathcal{U} \rangle. \quad (4.17)$$

This means that the transition probability is given by

$$P_{0 \rightarrow 1}(t) \approx \frac{2\tau}{m\hbar\omega_0} \int_0^\infty \rho(\omega) \cdot \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2} d\omega. \quad (4.18)$$

To evaluate this integral, do a change of variables to $x = (\omega_0 - \omega)t/2$ and integrate from $-\infty$ to ∞ to get that

$$P_{0 \rightarrow 1}(t) \approx \frac{\pi\tau\rho(\omega_0)}{m\hbar\omega_0} t. \quad (4.19)$$

The transition rate is therefore

$$R_{0 \rightarrow 1} \approx \frac{\pi\tau\rho(\omega_0)}{m\hbar\omega_0}. \quad (4.20)$$

4.3 ACOUSTIC ANALOGUES OF EINSTEIN'S A AND B COEFFICIENTS

We can now find the acoustic analogues of Einstein's A and B coefficients [2, 13], following the procedure he did, except our density of states now would be corresponding to the one for a one-dimensional string. The density of states (per unit length) for a one-dimensional string in the Debye model is

$$\mathcal{D} = \frac{1}{\pi c}, \quad (4.21)$$

so the spectral energy density is given by

$$\rho(\omega) = \frac{1}{\pi c} \cdot \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}, \quad (4.22)$$

where $\beta = 1/(k_B T)$ is the reciprocal of the system's fundamental temperature [16].

The transition rate therefore becomes

$$R_{0 \rightarrow 1} \approx \frac{\tau^2}{\sigma m c^3} \cdot \frac{1}{e^{\beta\hbar\omega_0} - 1}. \quad (4.23)$$

Analogous to the argument used by Einstein when he derived his coefficients [2, 13], the B coefficient is defined implicitly by

$$R_{0 \rightarrow 1} = B\rho(\omega_0). \quad (4.24)$$

This gives us that

$$B = \frac{\pi\tau}{m\hbar\omega_0}. \quad (4.25)$$

From Einstein's thermal equilibrium argument [2, 13], we conclude that the A coefficient, which gives the rate of spontaneous phonon emission, is given by

$$A = B \cdot \frac{\hbar\omega_0}{\pi c} = \frac{\tau}{mc}. \quad (4.26)$$

This is equal to twice the classical damping rate ν of the oscillating bead.

CHAPTER 5

THE QUANTUM MECHANICAL TREATMENT

In quantum mechanics, we fully treat both the bead and the string quantum mechanically. First, we would find the normal modes (eigenfunctions) of the wave's displacement function $u_s(x, t)$. For the wave equation

$$\frac{\partial^2 u_s}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u_s}{\partial t^2}, \quad (5.1)$$

an ansatz of the form

$$u_s(x, t) = w(x)e^{-i\omega t} \quad (5.2)$$

where $\omega = kc$ will give us the Helmholtz equation, namely

$$\frac{d^2 w}{dx^2} + k^2 w = 0. \quad (5.3)$$

This has solutions of the form

$$w(x) = A \sin(kx) + B \cos(kx) \quad (5.4)$$

where A and B are determined by the boundary conditions. As $w(0) = 0$, this means that $B = 0$, and since $w'(\ell) = 0$, this means that $k_n = (n + 1/2)\pi/\ell$ where $n \in \mathbb{N}_0$. We now can label each normal mode by the (positive) integer n as follows:

$$u_{sn}(x, t) = w_n(x)e^{-i\omega_n t} = A_k \sin(k_n x)e^{-i\omega_n t}. \quad (5.5)$$

We can also normalize the spatial dependence factor; the normalized eigenmodes are

$$u_{sn}(x, t) = \sqrt{\frac{2}{\ell}} \sin(k_n x)e^{-i\omega_n t}. \quad (5.6)$$

We are now ready to apply second quantization techniques (which are outlined in [14] and [15]) to this model.

5.1 QUANTIZATION OF THE STRING VIBRATIONS

Let Π denote the string's momentum density [14]; it's given by

$$\Pi \equiv \frac{\partial \mathcal{L}_s}{\partial \dot{u}} = \sigma \frac{\partial u_s}{\partial t}. \quad (5.7)$$

The Hamiltonian of the string is then given by

$$H_s = \int_0^\ell \left[\frac{\Pi^2}{2\sigma} + \frac{\tau}{2} \left(\frac{\partial u_s}{\partial x} \right)^2 \right] dx. \quad (5.8)$$

To quantize this Hamiltonian, expand Π in terms of the spatial eigenmodes in the following way:

$$\Pi = \sum_k P_k w_k(x) = \sqrt{\frac{2}{\ell}} \sum_k P_k \sin(kx). \quad (5.9)$$

We can also expand the displacement field in terms of the eigenmodes as well:

$$u_s(x, t) = \sum_k Q_k w_k(x) = \sqrt{\frac{2}{\ell}} \sum_k Q_k \sin(kx). \quad (5.10)$$

We can now exploit orthonormality to calculate the following integrals of interest:

$$\begin{aligned} \int_0^\ell \Pi^2 dx &= \int_0^\ell \sum_{k'} \sum_k P_k P_{k'} w_k(x) w_{k'}(x) dx = \sum_k P_k^2(t) \\ \int_0^\ell \left(\frac{\partial u_s}{\partial x} \right)^2 dx &= \frac{2}{\ell} \int_0^\ell \sum_{k'} \sum_k k k' Q_k Q_{k'} \cos(kx) \cos(k'x) dx = \sum_k k^2 Q_k^2 \end{aligned} \quad (5.11)$$

Our Hamiltonian now becomes

$$H_s = \sum_k \left(\frac{P_k^2}{2\sigma} + \frac{\tau k^2 Q_k^2}{2} \right) = \sum_k \hbar \omega_k \left(b_k^\dagger b_k + \frac{1}{2} \right). \quad (5.12)$$

Now, we have to factor the first expression in the following way:

$$\begin{aligned} \frac{P_k^2}{2\sigma} + \frac{\tau k^2 Q_k^2}{2} &= \hbar \omega_k \left(\frac{P_k^2}{2\sigma \hbar \omega_k} + \frac{\tau k^2 Q_k^2}{2 \hbar \omega_k} \right) \\ &= \hbar \omega_k \left[\left(\frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + k Q_k \sqrt{\frac{\tau}{2 \hbar \omega_k}} \right) \left(-\frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + k Q_k \sqrt{\frac{\tau}{2 \hbar \omega_k}} \right) + \frac{1}{2} \right] \\ &= \hbar \omega_k \left(b_k^\dagger b_k + \frac{1}{2} \right). \end{aligned} \quad (5.13)$$

(We also impose the canonical commutation rule $[P_k, Q_{k'}] = -i\hbar \delta_{kk'}$ in the process.)

But, we know that $c^2 = \omega_k^2/k^2 = \tau/\sigma$, so $\sigma \omega_k = \tau k^2/\omega_k$, and we get that

$$b_k^\dagger = \frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + k Q_k \sqrt{\frac{\tau}{2 \hbar \omega_k}} = \frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + Q_k \sqrt{\frac{\sigma \omega_k}{2 \hbar}} \quad (5.14)$$

$$b_k = -\frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + k Q_k \sqrt{\frac{\tau}{2 \hbar \omega_k}} = -\frac{i P_k}{\sqrt{2\sigma \hbar \omega_k}} + Q_k \sqrt{\frac{\sigma \omega_k}{2 \hbar}}, \quad (5.15)$$

which are called creation and annihilation operators of the string [14, 15]. (The reason these operators are called this way is because b_k^\dagger *creates* a transverse acoustic phonon on the string with wavenumber k , while b_k *destroys* a transverse acoustic phonon on the string with wavenumber k .)

We can also solve for the momentum and position coefficients, P_k and Q_k , respectively, in terms of b_k and b_k^\dagger from Eqs. (5.14) and (5.15):

$$P_k = i\sqrt{\frac{\sigma\hbar\omega_k}{2}}(b_k - b_k^\dagger) \quad (5.16)$$

$$Q_k = \sqrt{\frac{\hbar}{2\sigma\omega_k}}(b_k + b_k^\dagger). \quad (5.17)$$

5.2 FERMI'S GOLDEN RULE

We now fully quantize the Hamiltonian which is the sum of Eq. (4.3). Using the string displacement found in the previous section, and standard techniques [14, 15], the interaction Hamiltonian takes on the following form:

$$H_{\text{int}} = -\tau u_0 \sum_k \gamma_k Q_k, \quad (5.18)$$

where

$$\gamma_k = k\sqrt{\frac{2}{\ell}}. \quad (5.19)$$

Again, consider calculating the spontaneous emission rate from the bead, which initially starts out in its first excited state, $|1\rangle$, and decays to its ground state, $|0\rangle$. In this case, we can utilize Fermi's golden rule [2, 15] to calculate the bead's transition rate to first order.

We currently have that

$$\Gamma_{1 \rightarrow 0} = \frac{2\pi}{\hbar} \sum_k |\langle 0; 1_k | H_{\text{int}} | 1; 0 \rangle|^2 \delta(E_f - E_i + \hbar\omega_k), \quad (5.20)$$

where $\delta(x)$ is the **Dirac delta function**, E_i and E_f are the initial and final energies of the bead, respectively, and we have used the same initial conditions as in the semiclassical treatment [Eq. (4.9)], but in this case we need another index to label the string's eigenstates. We consider the string's phonons to start in the ground state, and they'll transition up into their first excited state as a result of the interaction.

From elementary quantum mechanics, we know that $E_f = \hbar\omega_0/2$ and $E_i = 3\hbar\omega_0/2$ in this case, and using the scaling property of the delta function, we get that

$$\Gamma_{1 \rightarrow 0} = \frac{2\pi}{\hbar^2} \sum_k |\langle 0; 1_k | H_{\text{int}} | 1; 0 \rangle|^2 \delta(\omega_k - \omega_0). \quad (5.21)$$

The matrix element $\langle 0; 1_k | H_{\text{int}} | 1; 0 \rangle$ can be evaluated by invoking the fact that

$$\langle 0; 1_k | H_{\text{int}} | 1; 0 \rangle = -\tau \sum_{k'} \gamma_{k'} \langle 1_k | Q_{k'} | 0 \rangle \langle 0 | u_0 | 1 \rangle, \quad (5.22)$$

but

$$\langle 0 | u_0 | 1 \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \quad (5.23)$$

and

$$\langle 1_k | Q_{k'} | 0 \rangle = \sqrt{\frac{\hbar}{2\sigma\omega_k}} \delta_{kk'} \quad (5.24)$$

(from selection rules), so

$$\langle 0; 1_k | H_{\text{int}} | 1; 0 \rangle = -\frac{\hbar\tau}{c} \sqrt{\frac{\omega_k}{2\sigma m\omega_0 \ell}}. \quad (5.25)$$

Plugging this newfound relation into Eq. (5.21) and taking the continuum limit, we

get that

$$\Gamma_{1 \rightarrow 0} = \frac{\pi\tau^2}{\sigma m \omega_0 \ell c^2} \int_0^{\omega_D} \mathcal{D}(\omega) \delta(\omega - \omega_0) \omega \, d\omega, \quad (5.26)$$

where ω_D is the Debye frequency of the system [16]. As the bead's natural oscillation frequency lies between zero and the Debye frequency, the integral above is certainly not zero. We use the shifting property of the delta function to find that

$$\Gamma_{1 \rightarrow 0} = \frac{\pi\tau^2 \mathcal{D}(\omega_0)}{\sigma m \ell c^2} = \frac{\tau}{mc} = 2\nu. \quad (5.27)$$

Therefore, the first-order quantum mechanical spontaneous emission rate is exactly equal to the first-order semiclassical spontaneous emission rate calculated previously.

We can also generalize these results to the case where the bead decays from state $|n\rangle$ to state $|n-1\rangle$, where n is a positive integer. Eq. (5.21) turns into

$$\Gamma_{n \rightarrow n-1} = \frac{2\pi}{\hbar^2} \sum_k |\langle n-1; 1_k | H_{\text{int}} | n; 0 \rangle|^2 \delta(\omega_k - \omega_0). \quad (5.28)$$

The new matrix element can be evaluated using the same procedure, we first split it into two matrix elements:

$$|\langle n-1; 1_k | H_{\text{int}} | n; 0 \rangle| = -\tau \sum_{k'} \gamma_{k'} \langle 1_k | Q_{k'} | 0 \rangle \langle n-1 | u_0 | n \rangle, \quad (5.29)$$

where $\langle 1_k | Q_{k'} | 0 \rangle$ was evaluated in Eq. (5.24), and

$$\langle n-1 | u_0 | n \rangle = \sqrt{\frac{n\hbar}{2m\omega_0}}. \quad (5.30)$$

Following the above procedure which was used to derive the spontaneous emission rate for the bead decaying from $|1\rangle$ to $|0\rangle$, we arrive at

$$\Gamma_{n \rightarrow n-1} = n \left(\frac{\tau}{mc} \right) = 2\nu \cdot n. \quad (5.31)$$

The Einstein A coefficient can now be labelled as

$$A_n = n \left(\frac{\tau}{mc} \right) = 2\nu \cdot n. \quad (5.32)$$

The time-averaged acoustic power radiated over an oscillation is therefore

$$\langle P_{\text{rad}} \rangle_n = n\hbar\omega_0 \cdot 2\nu = \left(E_{\text{initial}} - \frac{\hbar\omega_0}{2} \right) \cdot 2\nu. \quad (5.33)$$

Using Bohr's correspondence principle (where we take $n \gg 1$), we can simply ignore the zero-point energy of the oscillator and recover the classical result for the time-averaged acoustic power radiated over an oscillation, which is given by Eq. (3.55).

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this thesis, the classical Lamb model was reformulated in a Lagrangian framework. The use of Lagrangian mechanics yielded a set of coupled differential equations, which were decoupled and solved through the use of Laplace transforms. The exact expressions of the position functions for both the bead and string were found and plotted. While the bead showed familiar behaviour due to the nature of amplitude decrease, energy and power loss, it was interesting to note that due to the coupling condition, the string's displacement field had a Heaviside unit step function in its analytic expression, as a result of information delay from the finite wave speed. In the semiclassical treatment, the bead was chosen to be treated quantum mechanically but the string classically, and the interaction term was analyzed through the use of time-dependent perturbation theory. The bead's transition probability and rate were calculated, and the acoustic analogues of Einstein's A and B coefficients were found; furthermore, the A coefficient, which tells us the spontaneous transition rate of the bead, was equal to twice the bead's classical damping rate. In the quantum mechanical treatment, the total Hamiltonian was quantized through the use of creation and annihilation operators, and again the interaction term was treated using time-dependent perturbation theory. Fermi's golden rule was used to calculate the spontaneous emission rate of

the bead, and it agreed with the semiclassical result to first order. Furthermore, the quantum mechanical result for the time-averaged acoustic power radiated by the bead over an oscillation reduced to the classical result when Bohr’s correspondence principle was used. Overall, quantization of the one-dimensional Lamb model was performed successfully.

However, that is not to say that there is no more work to be done for this problem. While the classical and semiclassical treatments of the problem can be considered “complete,” the quantum mechanical treatment can be refined further. First, one can calculate the regime of validity of perturbation theory. Higher-order terms can be computed; by calculating the next-order contribution from the series

$$\left| \langle f | H_{\text{int}} | i \rangle + \sum_{\mu} \frac{\langle f | H_{\text{int}} | \mu \rangle \langle \mu | H_{\text{int}} | i \rangle}{E_i - E_{\mu}} + \sum_{\mu} \sum_{\mu'} \frac{\langle f | H_{\text{int}} | \mu' \rangle \langle \mu' | H_{\text{int}} | \mu \rangle \langle \mu | H_{\text{int}} | i \rangle}{(E_i - E_{\mu})(E_i - E_{\mu'})} + \dots \right|^2, \quad (6.1)$$

we can assess the accuracy of keeping only the first-order terms [17]. Here, μ and μ' are intermediate states in which the bead (may) transit to in the process of getting to its final states. These can be determined by selection rules. Furthermore, this would allow the calculation of transition rates to other states, which was not done in this thesis (only the simplest case where the bead decays from its first excited state was calculated).

A thermal dependence of the bead’s transition rates in this model was also not calculated in this thesis. With techniques used from thermal and statistical physics, one would be able to consider the string at some finite temperature T and then calculate the average number of phonons on the line. This would also allow one to determine an expression for the Debye frequency ω_D in terms of ω , ω_0 , and any other relevant quantities.

There is a critical damping rate ν_c which is the “cutoff” for the validity of perturbation theory. As long as $\nu \ll \nu_c$, perturbation theory gives well-described results.

An open question would be to study the regimes $\nu = \nu_c$ and $\nu > \nu_c$. By studying those two regimes, they may yield answers to deeper questions, like the possibility of dissipative phase transitions [18] and spontaneous symmetry breaking in the model. Note however, that these two regimes will have to be studied by means other than perturbation theory, as it breaks down in this case.

Finally, the generalization to two and three dimensions of this model can be studied. This would help model more realistic cases of the damped harmonic oscillators, such as the aforementioned quantum devices (which can be used for quantum sensing, for example), where the design of some devices is such that atoms vibrate in membranes and solids. Some work has already been done on the mentioned examples in recent years, such as [19] and [20], which study the models fully quantum mechanically.

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APPENDIX

When calculating the energy and power of the bead, the energy and power formulae required the calculation of $\dot{u}_0(t)$ and $\ddot{u}_0(t)$, and I mentioned a trick that I used to convert the various linear combinations of sines and cosines into a single sine or cosine with a phase shift. This is to simplify the algebra and avoid the product rule when calculating $\dot{u}_0(t)$ and $\ddot{u}_0(t)$. It is now only fair that I demonstrate the trick, which only uses elementary trigonometric functions, but a keen eye is needed.

Recall Eq. (3.39), which states that

$$u_0(t) = ae^{-\nu t} \left[\frac{\nu}{\omega} \sin(\omega t) + \cos(\omega t) \right]. \quad (6.2)$$

Now, consider rewriting it in the following way:

$$u_0(t) = ae^{-\nu t} \sqrt{1 + \frac{\nu^2}{\omega^2}} \left[\frac{\nu/\omega}{\sqrt{1 + \nu^2/\omega^2}} \sin(\omega t) + \frac{1}{\sqrt{1 + \nu^2/\omega^2}} \cos(\omega t) \right]. \quad (6.3)$$

But, we also have that

$$\sqrt{1 + \frac{\nu^2}{\omega^2}} = \frac{\sqrt{\omega^2 + \nu^2}}{\omega} = \frac{\omega_0}{\omega}, \quad (6.4)$$

so we technically have that

$$u_0(t) = \frac{a\omega_0}{\omega} e^{-\nu t} \left[\frac{\nu}{\omega_0} \sin(\omega t) + \frac{\omega}{\omega_0} \cos(\omega t) \right]. \quad (6.5)$$

Now, let $\sin(\phi) = \nu/\omega_0$ and $\cos(\phi) = \omega/\omega_0$, so we have that

$$u_0(t) = \frac{a\omega_0}{\omega} e^{-\nu t} [\sin(\phi) \sin(\omega t) + \cos(\phi) \cos(\omega t)] = U(t) \cos(\omega t - \phi), \quad (6.6)$$

with

$$\begin{aligned} U(t) &= \frac{a\omega_0}{\omega} e^{-\nu t}, \\ \phi &= \tan^{-1} \left(\frac{\nu}{\omega} \right) \end{aligned} \quad (6.7)$$

as claimed. ■

The same trick can be applied to $\dot{u}_0(t)$ and $\ddot{u}_0(t)$, by applying this procedure in the most convenient way possible, in order to reduce the amount of product rule calculations when one has to take the relevant time derivatives.