Edge Colored and Edge Ordered Graphs

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Per Wagenius

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Thesis Examination Committee:

Puck Rombach, Ph.D., Advisor
Christopher Landry, Ph.D., Chairperson
Spencer Backman, Ph.D.
Holger Hoock, DPhil, Dean of the Graduate College

Abstract

In this work, the current state of the field of edge-colored graphs is surveyed. A new concept of unshrinkable edge colorings is introduced which is useful for rainbow subgraph problems and interesting in its own right. This concept is analyzed in some depth. Building upon the linear edge ordering described in a recent work from Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, and Vizer, edge-ordering graphs with the cyclic group is introduced and some results are given on this and a related counting problem.

To my sister Hattie, who doesn't like math.

Acknowledgements

I am extremely grateful to my parents for their incredible support along my academic journey. I also want to thank my advisor Puck Rombach for all the inspiration, help, and encouragement along the way.

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Chapter 1

Introduction

1.1 What is a graph?

We will start by defining some terms commonly used in graph theory.

Definition 1 (Graph)**.** A graph G is a set of vertices and a set of tuples of vertices called edges. We write $V(G)$ for the set of vertices of G and $E(G)$ for the edge set of G. An edge $e = \{v_1, v_2\}$ should be thought of as connecting two vertices of G. We call the two vertices connected by e, v_1 and v_2, the endpoints of the edge e.

In this thesis we will restrict ourselves to simple graphs, which means that each edge must have two **distinct** endpoints and a pair of vertices can only have up to one edge connecting them.

Definition 2 (Order of a graph)**.** The order of a graph G is the number of vertices in G. We write $|G| = |V(G)|$ for the order of G.

Definition 3 (Size of a graph)**.** The size of a graph G is the number of edges in G. We write $||G|| = |E(G)|$ for the size of G.

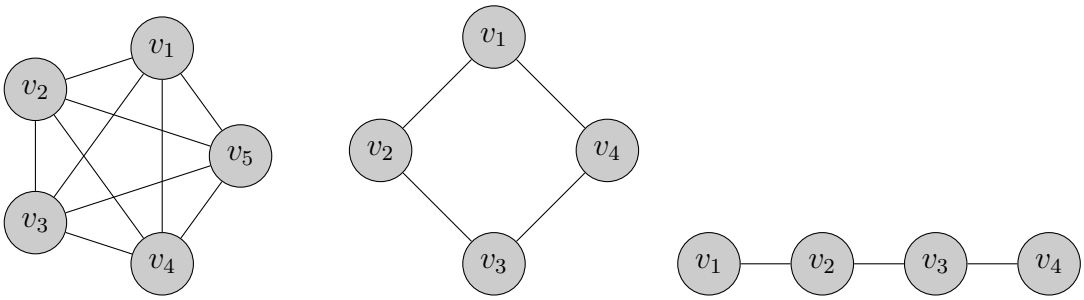
Definition 4 (Incidence)**.** We say that two edges e_1,e_2 of a graph G are incident if they share an endpoint, in other words $e_1 \cap e_2 = v$ for some $v \in V(G)$.
We say that an edge e and a vertex v are incident if v is an endpoint of e ($v \in e$).

Definition 5 (Degree)**.** The degree of a vertex $v \in V(G)$ is the number of edges incident to v, written $d(v)$. The max degree Δ is the degree of the vertex with the largest degree $\Delta = \max_{v \in (V(G))} d(v)$. The total degree of a graph G is the sum of the degrees of all of the vertices and the average degree of a graph is the total degree divided by the order.

Definition 6 (Subgraph)**.** We say that a graph H is the subgraph of a graph G if H is a graph, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It's important to note that we don't care about the precise names of the vertices themselves, we just care about the graph structure. For H to be a subgraph of G, H need only "look like" a part of G.

Frequently used graphs are often named. We will name some of these graphs and show them in Fig. 1.1. A path with order n is written P_n. The path P_n has vertex set $\{v_1, ..., v_n\}$ and edges $E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}\}$. We call v_1 and v_n the endpoints of the path P_n. A cycle C_n is a path P_n with an edge added to connect the endpoints of P_n. The complete graph on n vertices is written K_n. It has all possible edges, there is an edge connecting each pair of vertices. Thus, K_n has every graph of order less than or equal to n as a subgraph.

Example 1. We often think of paths and cycles as subgraphs. A Hamiltonian cycle of a graph G is a subgraph which is a cycle and uses all the vertices of G. In other words it is a cycle with the same order as G. A Hamiltonian path is defined similarly.



(a) Complete graph K_5 *(b) Cycle graph C_4.* *(c) Path graph P_4.*

Figure 1.1: Some common graphs

1.2 Edge Colorings

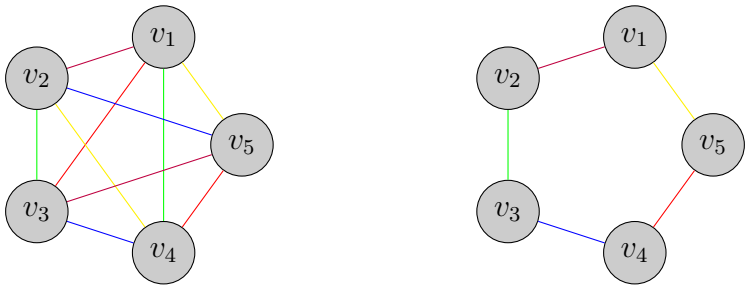
Now that we have defined some of the core concepts in graph theory, we can define an edge coloring of a graph.

Definition 7 (Edge Coloring)**.** A proper edge coloring of a graph G is a map from the edge set $E(G)$ to a set of "colors" so that no two incident edges have the same color. Using mathematical notation, a proper edge coloring of G is a function $c : E(G) \to \mathbb{N}$ with $e_1 \cap e_2 \neq \varnothing \implies c(e_1) \neq c(e_2) \forall e_1, e_2 \in E(G)$. We write "edge coloring" or simply "coloring" to refer to a proper edge coloring, and we write "colored graph" to refer to a graph equipped with an edge coloring.

We denote the set of n colors as $\{1, 2, ..., n-1, n\} = [n]$. We say that an edge coloring with t colors, also a "t-edge coloring," is a surjective map $E(G) \to [t]$.

Definition 8 (Chromatic Index)**.** We call the smallest number of colors needed to edge-color a graph the chromatic index or the edge chromatic number written $\chi'(G)$.

We call a coloring of G optimal if it uses the minimum number of colors ($\chi'(G)$ colors). We call a colored graph a rainbow graph if each edge has a different color. We are



(a) K_5 with an optimal edge coloring. *(b) a rainbow-C_5*

Figure 1.2: The optimal coloring of K_5 (a) contains a rainbow-C_5 (b)

particularly interested in rainbow subgraphs.

Definition 9. We say that a graph G colored by a function c contains a rainbow copy of H if there exists a subgraph H of G so that the edge set of H is colored with all different colors by c, the coloring of G. In other words, $|c[E(H)]| = |E(H)|$ or $c|_{E(H)}$ is one to one.

Example 2. An optimal coloring of K_5 uses $\chi'(K_5) = 5$ colors (Fig. 1.2a). This coloring of K_5 happens to be unique up to vertex and color label swapping. This coloring has a rainbow-Hamiltonian cycle, see Fig. 1.2.

The minimal graph containing all edges colored with a given color a is referred to as a's color class.

1.3 Rainbow Paths and Cycles in Complete Graphs

One of the foundational questions in the study of rainbow subgraphs is whether edge-colored graphs have long rainbow paths and cycles.

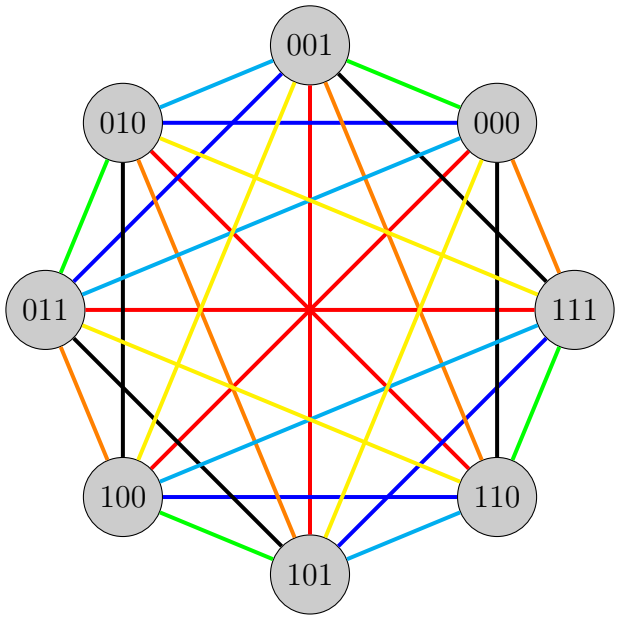


Figure 1.3: K_8 with edge coloring from Maamoun and Meyniel [1].

The path question was formally proposed by Hahn in 1980. Hahn conjectured that all edge colorings, including non-proper edge colorings, of the complete graph with more than 5 vertices have a rainbow Hamiltonian path, provided that each color is used at most $n/2$ times [2]. This was disproved by Maamoun and Meyniel four years later [1]. Their counterexample is coloring of K_{2^p} for any integer p so that the resulting colored graph does not have any Hamiltonian paths. Their counterexample can be though of as treating each vertex as a string of elements of $\mathbb{Z}/2\mathbb{Z}$ length p (there are $2^p = |G|$ such strings) and assigning each of $2^p - 1$ colors to each edge for which its incident vertices' strings differ by the same amount (Fig. 1.3). So in $K_{2^3=8}$, we would color the edge between the vertex labeled 101 and 111 the same as the edge between the vertices labeled 000 and 010, because both of those edges have endpoints labels which differ from each other by flipping their second element. If there was a rainbow-Hamiltonian path, it must use all the colors, and thus the label of the first and last endpoint must

differ by exactly the sum of all the different colors. But the sum of all the colors corresponds to not changing the vertex label between the endpoints of the path at all, in other words any sequence of $2^p - 1$ differently colored incident edges without a vertex of degree greater than 2 must be a cycle.

1.3.1 k-bounded Colorings Containing Rainbow-Hamiltonian Cycles

While the authors show that Hahn's conjecture is false, they affirm that the problem of investigating long rainbow paths and cycles within edge colored complete graphs is interesting and deserves further study. They ask a different question, based on the concept of k-bounded colorings.

Definition 10 (k-bounded colorings)**.** A coloring of K_n in which each color class is of size at most k as a k-bounded coloring.

Then, the authors note that for each complete graph K_n, there exists some k so that each k-bounded coloring contains a rainbow Hamiltonian path. A 1-bounded graph is rainbow by definition, so we focus on how big we can get k such that this rainbow Hamiltonian subgraph property still holds.

Question 1. *The authors ask how this largest bound k so that any k bounded coloring of K_n contains a rainbow Hamiltonian path grows with n.*

We note that if a certain coloring of K_n has a rainbow cycle of size l, then that coloring has a rainbow path of at least size $l - 1$, just by deleting one of the edges. So finding a rainbow Hamiltonian cycle will guarantee the existence of a rainbow

Hamiltonian path. As such, many authors only focus on when a k-bounded coloring implies the existence of a rainbow Hamiltonian cycle rather than such a path. We will present the results for cycles and then results for paths.

Hahn and Thomason [3] found a constant c so that each k-bounded colorings of K_n for $n \geq ck^3$ is guaranteed to have a rainbow-Hamilton cycle. In other words, we can ensure rainbow-Hamiltonian cycles by letting k grow no faster than $\sqrt[3]{n}$. The authors do this by considering the number of *crossings* and *switchings* in some long cycle C; the authors define a *switching* as a place where some consecutive vertices a, b in the cycle ($\{ab\} \in C$) have edges $\{av\}, \{vb\}$ going to a different vertex v and differing in color, so a switching gives the potential for a vertex to be added to our cycle, and *crossing* as way of switching the order of vertices along the cycle. The authors show that these switchings and crossings occur sufficiently frequently and that they can be used to augment the size of a cycle C. The authors then conjecture that letting k grow no faster than n would be sufficient for the existence of rainbow-Hamiltonian cycles, though, and direct future work towards showing the existence or nonexistence of this linear bound instead of using their method and improving the constant factor.

Rödl and Winkler in 1984 privately corresponded to show that letting $n \geq ck^2$ is sufficient instead of the bound in [3] of $n \geq ck^3$ [4]. Calling a coloring containing a rainbow-Hamiltonian cycle *good*, Frieze and Reed in [4] go on to show that letting k grow with $\frac{n}{\ln n}$ is sufficient for each k bounded coloring of K_n to be good for large values of n, in their words there exists some large constant A such that if n is sufficiently large and $k \leq \lceil \frac{n}{A \ln n} \rceil$, then any k-bounded coloring of K_n contains a rainbow Hamilton cycle.

The constant bound conjectured by Hahn and Thomassen [3] was indeed achieved

by Michael Albert, Alan Frieze and Bruce Reed in 1995. They used a probabilistic method in addition to some clever ways of making a cycle longer to show that if n is large enough, any coloring of K_n which is k bounded for some $k \leq \lceil n/32 \rceil$ is good, i.e. it contains a rainbow Hamiltonian cycle. [5]

1.3.2 Long Rainbow Cycles in K_n

To further investigate the properties of proper colorings of K_n, many authors have studied different lower bounds for the largest rainbow path or rainbow cycle in K_n. The existence of Maamoun and Meyniel's counterexample to Hahn's conjecture is thought to be quite special, and almost-Hamiltonian rainbow paths are thought to exist in every coloring of a complete graph [6]. We will consider the lower bounds for the largest rainbow-cycle or rainbow-path in K_n. We will review the associated results for rainbow cycles, then for rainbow paths.

According to [7], any coloring of K_n contains a rainbow cycle with at least $n/2-1$ colors (thus $n/2-1$ edges). The authors do this by contradiction. They fix a maximum size rainbow cycle C within a coloring of K_n and suppose that $|E(C)| < n/2-1$ edges. Using two clever ways of adding edges to C, they show that C can be expanded. This bound is further improved by [6], where the authors construct more possible detours along a cycle that would increase its length whilst maintaining its polychromaticism. These more complicated detours allow the authors to prove that any coloring of K_n for large enough n contains a rainbow cycle of almost $\frac{4n}{7}$ colors. This bound is improved by [8] to show that any coloring of K_n has rainbow cycles as big as $n - O(n^{\frac{3}{4}})$. The method used by Alon et al. involves finding connecting rainbow path forests, sets of disjoint rainbow paths, combined with a result showing that random subgraphs

of properly edge - coloured complete graph have distributions similar those of truly random graphs with the same densities.

1.3.3 Long Rainbow Paths in K_n

As mentioned earlier, a rainbow-cycle with l edges has a rainbow-path with $n-1$ edges as a subgraphs. Some of our results for long paths come trivially from the results on cycles described earlier, in this section we focus our attention on other results on the existence of long rainbow path. In 2010, the authors of [9] showed that each proper edge coloring of K_n has a rainbow path of length $\frac{2n+1}{3}$. They did this by designing a procedure to augment the length of a rainbow cycle and show that this process does not terminate before reaching a path of at least $\frac{2n+1}{3}$. This bound was improved by [10] and [11] to a bound of $(\frac{3}{4} - o(1))n$ in two different ways. However, the best bound is still equivalent to that in the cycle case, by [8] any coloring of K_n has rainbow cycles as big as $n - O(n^{\frac{3}{4}})$ so there exists rainbow paths of length $n - O(n^{\frac{3}{4}}) - 1$.

1.4 The Extremal Number

In extremal graph theory, we study how by starting with a vertex set and adding edges, we eventually force some structures to be introduced into our graph. For example, if we add edges to a graph with a set of n vertices and no edges, we eventually generate a triangle within our graph. We will define this notion rigorously and then explore this specific problem as an example.

Definition 11 (The extremal number)**.** We consider the class of graphs which do not

have a given graph H as a subgraph, and we ask what is the largest size (number of edges) in such a graph with a given order (number of vertices). We write $\mathrm{ex}(n, H)$ for this number which depends on its two parameters, H, the forbidden subgraph and n, the fixed number of vertices.
We often want to exclude a collection of forbidden subgraphs $\mathcal{H}$. In this case we write $\mathrm{ex}(n, \mathcal{H})$ for the largest number of edges a graph on n vertices can have whilst not having any of the graphs within $\mathcal{H}$ as a subgraph.

We will give an example for one of the first problems answered in this field, which is that of "triangle-free" graphs (graphs which do not have K_3 as a subgraph). That is $\mathrm{ex}(n, K_3)$, the maximum number of edges a graph on n vertices can have without having K_3 as a subgraph.

Theorem 1 (Mantel (1907) [12])**.** $\mathrm{ex}(n, K_3) \leq \lfloor \frac{n^2}{4} \rfloor$. *That is, any triangle-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.*

To find a construction which nears this bound on n vertices, split n into a set of $\lfloor n/2 \rfloor$ vertices and a set of $\lceil n/2 \rceil$ vertices and add every edge between the two sets. The $n = 8$ case is drawn in figure 1.4. This construction is called the complete bipartite graph $K_{n/2,n/2}$ and also the Turán graph $T(n, 2)$ (the second parameter means we have two nearly even groups). If $2|n$, this graph has every edge between a group of $n/2$ vertices and another group of $n/2$ vertices and thus has $n^2/4$ edges, achieving the upper bound in (Thm. 1). In general, the Turán graph $T(n, r)$ splits n vertices into r nearly even groups (all groups can only be made the same size if $r|n$) and includes all edges between vertices of different groups. The Turán graph is a good segue into more extremal graph theory as it was named after the mathematician who

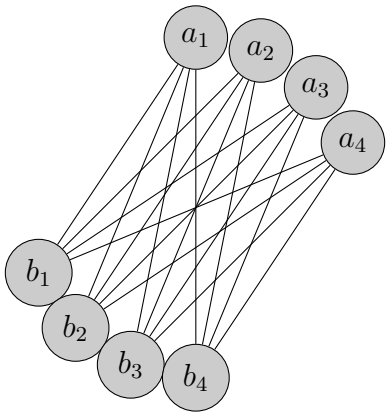


Figure 1.4: Turan graph $T(8,2)$.

started the systematic study of extremal graph theory problems [13]. In [14], Turán found the general extremal number $\mathrm{ex}(n, K_{r+1})$ for $r \geq 2$, see (Thm. 2).

Theorem 2 (Turán (1941))**.** *The Turán graph $T(n,r)$ is the best graph for avoiding K_{r+1}. That is* $\mathrm{ex}(n, K_{r+1}) \leq |E(T(n,r)|$

After finding the extremal number for complete graphs, the next foundation problem in extremal graph theory is the extremal number of the Turán graph $T(n,r)$ when n is a multiple of r. In 1946, Erdös and Stone proved that the graph $T(tr,r)$ has extremal number $\mathrm{ex}(n, T(tr,r)) = (\frac{r-2}{r-1} + o(1))\binom{n}{2}$ [15]. This foundational theorem is so valuable since in can give us great insight into graphs which are not Turán based on the chromatic number $\chi(H)$ which is the minimum number of colors in a proper vertex coloring of H:

Theorem 3 (Erdoš-Stone-Simonovits Theorem)**.** *If H is a graph with chromatic number $\chi(H) \geq 2$, then* $\mathrm{ex}(n,H) = (1 - \frac{1}{\chi(H)-1} + o(1))\binom{n}{2}$.

Extremal graph theory has been a rich field since its conception. This overview is only skimming the surface, an in depth history like [13] by Füredi and Simonovits

has much more information.

1.4.1 Forbidden Vertex and Edge Ordered Graphs

The field of extremal graph theory has also been extended to questions with vertex-ordered [16] [17] and very recently edge-ordered subgraphs [18]. In the vertex-ordered papers, the authors consider vertex ordered graphs and extend the notion of subgraph to respect vertex order. Then, they look at a new extremal number which takes into account this new notion of subgraphs. If H is a vertex ordered graph, then we can define a new extremal number which represents the largest size a graph G can have on n vertices such that there exists some vertex ordering of G such that G equipped with this ordering does not contain any copies of the vertex-ordered H with containment defined by this stricter notion of subgraphs.

Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer in [18] define a similar notion for edge ordered graphs. We will explore their paper further, as well as generate some results for graphs with edges ordered in a different manner in **Chapter 3**.

Chapter 2

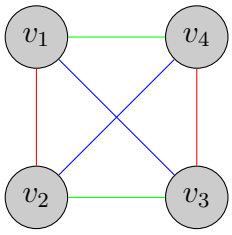
Unshrinkable Graphs

2.1 Motivation

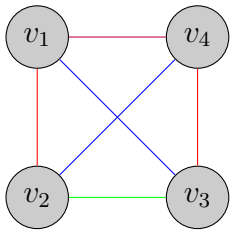
While investigating which colorings of the complete graph K_n are free of some rainbow subgraph H, we find that the best colorings are not necessarily optimal colorings. So the "search space" for trying to find a rainbow-H-free coloring of K_n (or a general graph G) is quite large, we can't just look at the optimal colorings of K_n, the naive approach would involve looking at all the colorings of K_n. In Subsection 1.3.1, we reviewed results that prove that for a given K_n, there exists some k so that each k-bounded coloring cannot avoid some rainbow Hamiltonian cycle. Thus, k-bounded colorings can be removed from our search space if H is some subset of a Hamiltonian cycle. However, we would like to narrow our search space further.

Question 2. *If we want to check whether a graph G can be colored to avoid a rainbow H, What subset of the edge-colorings can we look at?*

For our main observation, let's suppose we have some t-coloring c of K_n and we



(a) c, an edge coloring of K_4.



(b) c', a coloring made from c by changing the edge $\{v_1, v_4\}$ to a new color.

Figure 2.1: Two colorings c and c' of K_4

take some color class of c and change some of the edges in that color class to be colored with a new color $t + 1$ not used in c. Let's call this new colorings c', as in Fig. 2.1. We observe that this new coloring c' cannot be better at avoiding a rainbow subgraph than c. That is to say, if c contains a rainbow subgraph H, so to must c'. This is because every subgraph of G has the same or more colors under c' than under c, and c' would have to have less colors along H to avoid H when c does not. To return to our question, we want to exclude graphs like c' from our search space. So we will rigorously define an action that would get us from a coloring like c' to a coloring like c, and colorings upon which we can perform this action will be removed from the search space.

2.2 Definitions

2.2.1 Shrinking

We want to formalize a map that would map from c' to c in Fig 2.1. The map that we want would color both the green and the purple color class with the color green. If we map the color purple to the color green, and each other color of c' to itself with

the map s, then $s \circ c'$ is equivalent to c. We will call the map s a shrinking from c' to c. Note that this map s corresponds to a map from $[4] \to [3]$ if the colors red, blue, green, and purple correspond to the numbers 1, 2, 3, and 4.

Definition 12. We define a shrinking of a proper edge coloring as switching all the edges of a specific color to that of another color within our coloring. In mathematical terms, a shrinking of a t-edge-coloring c is a map $s : [t] \to [t-1]$ so that $s \circ c$ is a proper edge coloring.

2.2.2 Unshrinkable graphs and $X(G)$

We call a proper edge colored graph "unshrinkable" if its coloring does not have a shrinking.

The lower bound for the number of colors in a unshrinkable coloring is the chromatic index, so we concern ourselves with the upper bound for such a coloring which we write $X(G)$. That is, all unshrinkable k-colorings of a given graph G have $\chi'(G) \leq k \leq X(G)$.

2.3 Preliminary Results

We have that a proper edge colored graph is unshrinkable iff each color class intersects each other color class. This means that for every pair of two colors a, b there exists a vertex incident to both an edge colored a and an edge colored b.

An unshrinkable coloring has at most $\max\{\Delta(G), 3\}$ colors which only appear once, as if more than 3 colors which only appear once are present in a graph they must all be incident to each other, and this is only possible for four or more colors if they are

each incident to the same vertex.

We know that each pair of color classed contain incident edges. We can think of each vertex as an opportunity for a number of color classes to intersect and use this to get a condition for unshrinkable graphs.

Proposition 4. *For a k-edge coloring of G to be unshrinkable, we have*

$$\binom{k}{2} \leq \sum_{v \in V(G)} \binom{d(v)}{2}.$$

Proof. Each vertex $v \in V(G)$ as incident to $\binom{d(V)}{2}$ pairs of edges. Each of these pairs of edges is an "opportunity" for two color classes to intersect. A k-edge-coloring has a $\binom{k}{2}$ pairs of color classes, there must be at least this many "opportunities" in G. □

2.4 $X(G)$ FOR CYCLES

We will now find $X(G)$ precisely for G a general cycle.

Theorem 5. *For a general n, we have that $X(C_n) \leq \frac{1+\sqrt{1+8n}}{2}$. We have equality when $n = \frac{k(k-1)}{2}$ for some odd integer k.*

Proof. For the first part, using Prop. 4 we get that since all the vertices in C_n are of degree 2, we have $\binom{X(C_n)}{2} \leq \sum_{v \in V(C_n)} \binom{d(v)}{2} = n$. Simple rearranging $X(C_n) \leq \frac{1+\sqrt{1+8n}}{2}$.

For the upper bound, let $n = \frac{k(k-1)}{2}$ for some odd integer k. Let the vertices of K_k be labeled with the k colors we wish to use to color C_n. Then fix a Eulerian cycle of K_k, we know that K_k has a Eulerian cycle since each vertex of K_k is of degree $k-1$ which is even by supposition. Color the edges e_1 through e_n of our cycle C_n by the label

of the vertices we see along the Eulerian circuit. Now each edge from K_k is incident to two vertices labeled with different colors, and that edge corresponds to a vertex in C_n which sees the same color pair as the edge in K_k. Since each vertex pair in K_k is connected by an edge in K_k on our Eulerian tour, each two-color pair is incident to the corresponding vertex in C_n thus each color pair is incident to each other and C_n is unshrinkable, as desired. □

2.5 $X(G)$ FOR COMPLETE GRAPHS

Let's consider the case where G is some complete graph K_n. We want to know how many colors can we use in some unshrinkable coloring of K_n, $X(K_n)$. We will showcase a construction of an unshrinkable coloring of K_n for a lower bound of $X(K_n)$ and a proof for an upper bound of $X(K_n)$.

Theorem 6 (Lower bound for $X(K_n)$)**.** *We have that* $X(K_n) \geq 3n - o(\sqrt{n})$.

Proof. Let's suppose n is a triangle number, that is $n = 1 + 2 + ... + x = \frac{x(x+1)}{2}$. Additionally, suppose $2|x$, $x \geq 4$ and $2|n$. Now we can think of the vertex set of K_n as $x + 1$ sets each of x vertices with each set intersecting each other set at exactly one distinct vertex. Then optimally edge color the subgraph of K_n induced by each distinct set of x vertices with $x - 1$ different colors, which we can do since each such subgraph is a copy of K_x and x is even. The coloring so far has $(x - 1)(x + 1)$ colors. We can then colors the rest of the edges of K_n with colors incident to at least $n-x+1$ vertices. By a total degree argument, we have that $x^2/2-x/2-7/2-3/2x \geq x^2/2 - x/2 - 3$ colors can be added in this way, yielding a total of $3x^2/2 - x/2 - 4$ colors which, since $x^2/2 \approx n$ is $3n - o(\sqrt{n})$.

To show this coloring is unshrinkable, fix two arbitrary colors a, b and we will show that their color classes intersect. We have two sets of colors, the $(x-1)(x+1)$ colors we used in the first step of our coloring and the color with color class incident to $n-x+1$ vertices. Break into cases based on which sets a, b belong to. Case: a, b belong to the set of $(x-1)(x+1)$ colors. Then by construction a and b either both color the same K_x or they color two different K_x which intersect at a vertex. Either way, a and b color edges incident to each other. Case: a, b belong to different sets. Then without loss of generality suppose a is of the first set and b is of the second set. Then a has a color class incident to x vertices and b has a color class incident to $n-x+1$ vertices. Since there are only n vertices, these two color classes intersect at some vertex v and thus there is an edge of color b and an edge of color a both incident to v. Case: Both a, b are of the latter set of colors, i.e. their color classes have $n-x+1$ vertices. Since $n-x+1$ vertices is more than half of the vertices, the color classes intersect. □

So we have a lower bound for the number of colors we can use in an unshrinkable coloring of K_n. We also find an upper bound of $X(K_n)$.

Theorem 7 (Upper bound for $X(K_n)$)**.** *$X(K_n)$ grows no faster than $n^{3/2}$.*

Proof. Suppose n is a square number divisible by four. Fix an arbitrary unshrinkable coloring c of K_n. Let $x = \sqrt{n}/2$, and note x is an integer. Now either all of the color classes of c are of size x or more, or the smallest color class of c is of size at most $x-1$. In the first case, there are $||K_n|| = \frac{(n)(n-1)}{2}$ edges of K_n split into color classes of size at least x, and thus there at most $||K_n||/x = \frac{(n)(n-1)}{2x} = \frac{2(n)(n-1)}{2\sqrt{n}} = \sqrt{n}(n-1)$ colors. In the latter case, fix the smallest color class. It has at most $x-1$ edges and

thus at most $2(x-1)$ vertices. Since each other color class of c must have an edge incident to one of the vertices of this smallest color class, the number of other colors is limited by the sum of the degrees of each vertex of K_n adjacent to some edge of the smallest color class. Each of these $2(x-1)$ vertices is of degree $n-1$, and thus it can have $n-2$ other edges incident for a total of $2(x-1)(n-2)$ other colors and $2(x-1)(n-2)+1 = (\sqrt{n}-2)(n-2)+1$ total colors. In either case, there are less than $n\sqrt{n} = n^{3/2}$ colors, as desired. When n is not a square number divisible by four, we simply round x to the nearest integer and get a similar bound. □

2.6 Unshrinkable colorings of K_n for low n

We claimed at the start of this chapter that there are examples where no optimal colorings avoid some rainbow subgraph H, but some non-optimal unshrinkable colorings can avoid a rainbow H. We will showcase some of these colorings by finding the unshrinkable colorings of K_n for $n = 4, 5$, and many colorings of K_6, and showing which avoid some rainbow paths and cycles. This also serves as evidence to our claim that the "search space" of unshrinkable colorings of K_n is much more approachable then the total colorings. Since these colorings for graphs of small order were found by hand, we believe that similar methods implemented on a computer could take advantage of this reduction of search space for larger n. Table 2.1 illustrates our results. We will show the other results in the next sections.

For a note on what it means to be a "unique" coloring, we say that two colorings are isomorphic if swapping the vertex labels and swapping the color labels can make two

G	number colors of colors	partition of ||G|| into color class size	unique colorings	avoids which rainbow subgraphs
K_4	3	6=2+2+2	1	C_3, P_3
K_5	5	10=2+2+2+2+2	1	
K_5	6	10=2+2+2+2+1+1	1	
K_5	7	10=2+2+2+1+1+1+1	1	C_5
K_6	5	15=3+3+3+3+3	1	C_6, P_6
K_6	6	15=3+3+3+2+2+2	3 1	C_6
K_6	7	15=2*3+4*2+1	2	
K_6	7	15=3+6*2	?	?
K_6	8	15=3+5*2+2*1	?	?
K_6	8	15=7*2+1	4	

Table 2.1: Unshrinkable colorings of K_4, K_5, and K_6

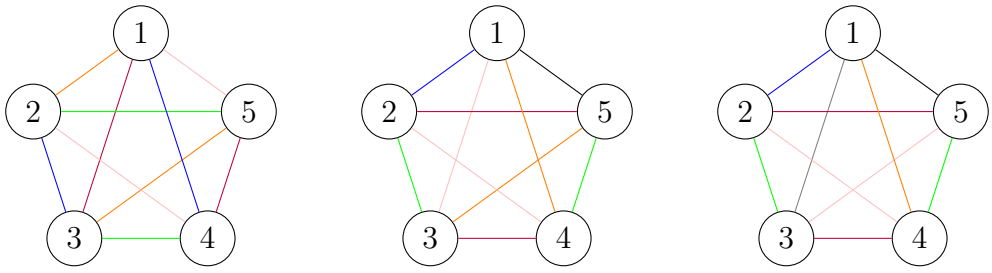
graphs identical. We won't count two isomorphic graphs more than once.

The $n = 4$ case is trivial, the optimal proper edge coloring with 3 colors is the only unshrinkable coloring. We will show the $n = 5$ and $n = 6$ cases in the following sections.

2.6.1 Unshrinkable colorings of K_5

There are three unshrinkable colorings of K_5. They have 5, 6, and 7 colors and are given by the following three matrices or equivalently by the drawings in Figure 2.2.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 1 & 0 & 3 & 5 & 4 \\ 2 & 3 & 0 & 4 & 1 \\ 3 & 5 & 4 & 0 & 2 \\ 5 & 4 & 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 5 & 2 \\ 1 & 0 & 6 & 3 & 4 \\ 3 & 6 & 0 & 4 & 5 \\ 5 & 3 & 4 & 0 & 6 \\ 2 & 4 & 5 & 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 7 & 6 & 5 \\ 2 & 7 & 0 & 5 & 6 \\ 3 & 6 & 5 & 0 & 7 \\ 4 & 5 & 6 & 7 & 0 \end{bmatrix}$$



(a) 5-coloring *(b) 6-coloring* *(c) 7-coloring*

Figure 2.2: Colorings of K_5

Proposition 8. *The aforementioned colorings are the only unshrinkable colorings of K_5.*

Proof. We consider the partitions of 10, the number of edges of K_5, into different sizes of colors. We can think of the minimal edge coloring with 5 colors as corresponding to the partition 10=2+2+2+2+2. It is already known that there is only one optimal edge coloring of K_5. Any proper coloring of K_5 with 6 colors corresponds to the partition $10 = 2+2+2+2+1+1$. It suffices to show that this parition corresponds to a single possible coloring. The two colors classes of size one must share a vertex, call this shared vertex v_1 and the other vertices contained in colors classes of size one v_2 and v_3. Then the color class containing the edge which isn't incident to any of the color classes of size 1, $\{v_4, v_5\}$, must also have the edge $\{v_2, v_3\}$ for its color class to intersect both color classes from colors of size 1. By a degree argument another color missed the vertex 1, without loss of generality let that color have edges $\{v_2, v_4\}$ and then it must have the edge $\{v_3, v_5\}$. Then the last two colors must each have one edge from $\{v_1, v_4\}$ or $\{v_1, v_5\}$ and one edge from $\{v_2, v_5\}$ or $\{v_3, v_4\}$. We only have one choice of pairings of those edges into color classes in order to keep the coloring proper. So This is the only unshrinkable coloring with 6 colors. The last partition to

investigate is 10=2+2+2+1+1+1+1. Since all four of the 1-edge colors must share a vertex, we are left with a subgraph of K_4, which can only be colored properly in one way by colors of size 2.
Using more colors would imply we would be using more than Δ colors with only one edge, which contradicts one of our preliminary results. □

In each coloring, We can find rainbow paths of all sizes (up to Hamiltonian paths) by inspection. However, while the optimal 5-coloring and the 6-coloring contain cycles of all sizes (up to Hamiltonian cycles), the last coloring has only rainbow subgraphs of C_3 and C_4. For a proof that this coloring contains no rainbow Hamiltonian cycle, note that this coloring can be constructed by optimally coloring K_4 and then adding a vertex v (the first column/row in the matrix) so that all the edges incident to that vertex are new colors. Now a Hamiltonian cycle can be thought of as starting at v, then hitting all of the vertices in the subset of K_5 colored by K_4, then returning to v. In other words, a Hamiltonian cycle would be a Hamiltonian path in K_4 joined at both ends by the vertex v. But any Hamiltonian path in K_4 with our optimal coloring is not rainbow, so any Hamiltonian cycle in this coloring of K_5 is not rainbow.

Remark. We note that this construction can be generalized. If there exists a coloring of K_n with no rainbow Hamiltonian paths, then we can construct a rainbow Hamiltonian cycle-free coloring of K_{n+1} by adding a vertex v and adding each edges vv' for $v' \in K_n$ with a color not previously used. So we can color each K_{2^k+1} in this way using Maamoun and Meyniel's construction [1].

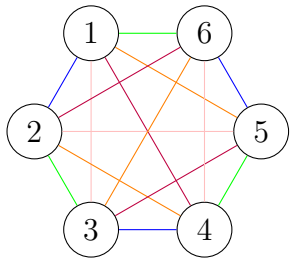
2.6.2 Unshrinkable colorings of K_6

We associate each coloring of K_6 to a partition of the total edges 15 into the sum of the sizes of each of the color classes in order to more easily think about these colorings. It is well known that the optimal proper coloring K_6 is unique. Since it uses 5 colors of size 3, it corresponds with the partition 15=3+3+3+3+3 of $||K_6||$ into sizes of color classes. We show the adjacency matrix and a drawing of this coloring in Figure 2.3.

It has no rainbow Hamiltonian cycle (it does not have enough colors) nor a rainbow

$$\begin{bmatrix} 0 & 1 & 3 & 4 & 5 & 2 \\ 1 & 0 & 2 & 5 & 3 & 4 \\ 3 & 2 & 0 & 1 & 4 & 5 \\ 4 & 5 & 1 & 0 & 2 & 3 \\ 5 & 3 & 4 & 2 & 0 & 1 \\ 2 & 4 & 5 & 3 & 1 & 0 \end{bmatrix}$$

(a) Adjacency matrix



(b) Coloring of K_6

Figure 2.3: Adjacency matrix and corresponding coloring

Hamiltonian path. By inspection, we can see that is contains all non-Hamiltonian rainbow paths and cycles.

For non optimal colorings, we will look through the other partitions of 15 made up of integers of at most 3. We will start by showing that many partitions do not have any associated unshrinkable colorings, and then we will show that those which do are precisely those noted in Table 2.1 by running thorough those partitions.

First, we claim that partitions 15=3+3+3+3+2+1 and 3+3+3+3+1+1+1 yield no unshrinkable colorings. Since each color class of size three is a prefect matching, the

4 color classes of size 3 combine to make a 4-regular graph. So the remaining color classes must make a 1-regular graph, and thus they don't intersect, so they aren't unshrinkable.

Next we check partitions with 3 color classes of size 3. We claim that $15 = 3 + 3 + 3 + 2 + 2 + 2$ is the only such partition which corresponds to unshrinkable colorings. Consider the partition $15 = 3 + 3 + 3 + 2 + 2 + 1 + 1$. For the same reason as the previous partition $15 = 3+3+3+2+2+2$, the graph made up by the colors of size less than 3 must be 2-regular. But for a 2-regular graph, each edge is only incident to two other edges. So a color class of size one can only be incident to two edges in color classes of size less than 3, but it must be incident to an edge from each color class of size less than 3, a contradiction since there are more than 2 color classes with size less than 3. Similar reasoning prevents partitions 15=3+3+3+2+1+1+1+1 and 15=3+3+3+1+1+1+1+1+1.

When we have exactly 2 color classes of size 3, we claim that 15=3+3+2+2+2+2+1 is the only partition yielding an unshrinkable coloring. We use the same reasoning, in this case we need the colorings of size less than 3 to generate a 3-regular graph. So each edge in a color class of size 1 must be incident to each other color class of size less than 3. For a 2-regular graph, each edge is only incident to four other edges. If there is a color class of size 1, there must be no more than 4 more colors in classes of size less than 3, which only occurs in the first partition.

When we are interested in colorings with one color class of size 3 we recall our earlier result that a coloring on any graph G cannot have more than the max degree $\Delta(G)$ color classes of size 1, which is 5 in this case. So when we have one color class of size 3, it suffices to only show that the partition 15=3+4*2+4*1 yields no unshrinkable

colorings. The 4 color classes of size 1 must share a vertex, thus making a star with 4 leaves. So the color classes of size 2 with an edge incident to the vertex outside this star cannot have only one other edge and thus cannot all intersect with all the colors of size 1. For the partitions with no color classes of size 3, we note that if there are 3 color classes of size 1, the color classes of size 1 must form a star with 3 leaves or a triangle. In either class, there is an edge not adjacent to any of these color classes, the color class of size 2 containing this edge cannot have its other edge incident to each of the color classes of size 3. Instead, if there are 5 color classes of size 1, they form a star with 5 leaves, and each color of size 2 can only intersect with 4 of the color classes of size 1, again not unshrinkable.

So we have shown that the partitions in the table are the only ones yielding unshrinkable colorings, next is to find what those unshrinkable colorings are, and whether they are free of long rainbow paths and cycles.

We will go row by row through the table.

Proposition 9. *There are 4 unique 6-colorings of K_6 associated to the partition 15=3+3+3+2+2+2. These are the four colorings shown by Figure 2.4.*

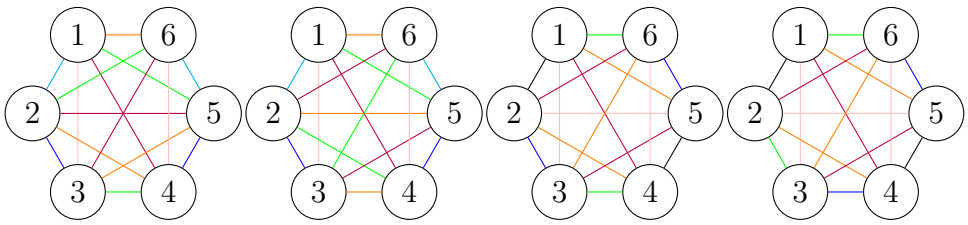
Proof. In all such colorings, the three colors of size 3 are perfect matchings so combined they generate a 3-regular subgraph graph. Thus, the subgraph of K_6 colored by the colors of size less than 3 must also be regular, in this case 2-regular. As a 2 regular subgraph, they must be a forest of cycles, either two copies of C_3 or one copy of C_6. We split into each of these cases.

Case: Our colors of non-maximal size form two copies of C_3. Then WLOG let the two cycles be the cycles v_1, v_2, v_3 and v_4, v_5, v_6 so that the colorings of size two look like those cycles in Figure 2.4 (a) and (b).

Now we have two different ways of filling the remaining edges of our K_6 graph with the maximal-size edges. Either let one of our three colors go in between each of the three vertex pairs which see the same colors from our coloring of $2C_3$ by colors of size 2, as in (b) or instead let those three edges be uniquely colored from the remaining colors (a). In either case the rest of the coloring follows uniquely.

Case: Our colors of non-maximal size form C_6. There are two proper colorings of C_6 with three colors each of size 2, either take a cycle with each color having pairs exactly opposite from each other within the cycle (c) or the other option only one color correspond to a pair of edges opposite from each other in the cycle (d). In either case where the graph is made up of this C_6 coloring and other edges, without loss of generality we can take the cycle to be around the outside of our K_6, then for the remaining colors of size 3 we must take one of the diagonal edges across the cycle to be each different color and then the other edge colors are uniquely determined.

So we have 4 unique colorings of K_6 with 6 colors corresponding to the partition of the edges of K_6 into 3 colors of size 3 and 3 colors of size two. □



(a) Color classes of size 2 form 2 C_3 *(b) Color classes of size 2 form 2 C_3* *(c) Color classes of size 2 form C_6* *(d) Color classes of size form C_6*

Figure 2.4: Colorings corresponding to the partition 3+3+3+2+2+2

Among these 4 colorings, we claim that (a),(b), and (d) avoid a rainbow Hamiltonian cycle and none of them avoid any other rainbow paths or cycles. To show that colorings (a) and (b) avoid a rainbow Hamiltonian cycle, we'll call the subset

of vertices $\{v_1, v_2, v_3\}$ A and call B the vertex subset $\{v_4, v_5, v_6\}$. We note that the colors orange, green, and purple have edges which only go between the subsets A and B, and the other 3 colors no edges going between these subsets. A Hamiltonian cycle would have to have an even number of edges going between A and B in order for it to be a cycle, but a rainbow Hamiltonian cycle would need to use each color once and thus have 3 edges going between A and B, a contradiction. To show that (d) avoids a rainbow Hamiltonian cycle, we'll call the edges $\{v_1, v_4\}$,$\{v_2, v_5\}$, and $\{v_3, v_6\}$ diagonals and note that all the diagonals are different colors. So a rainbow Hamiltonian cycle in this graph uses either 3,2,1, or 0 of these diagonals. Each case has no rainbow cycles. There are only 2 Hamiltonian cycles with all the diagonals, and none of those are rainbow by inspection. Using 2 diagonals means that we would need to use one of the non-diagonal edges of the third color.

Proposition 10. *The colorings given in Figure 2.5 are the only two unshrinkable colorings corresponding to the partition 15=3+3+2+2+2+2+1. Each of these colorings contains a rainbow Hamiltonian cycle.*

Proof. We start by noticing that each color class of size 3 is a perfect matching they must combine to a 2-regular graph. This must be either $2C_3$ or C_6 but we cannot properly color $2C_3$ with 2 colors so the color classes of size 3 must generate a C_6 with alternating colors in order to be proper. WLOG suppose this cycle is given by the vertex labels $v_1, v_2, v_5, v_6, v_4, v_3$. Now WLOG color the edge $\{v_1, v_6\}$ by the color only used once. Now there are only two ways of putting in the colors used twice in a way that each color has an edge incident to the edge $\{v_1, v_6\}$, which we need for the coloring to be unshrinkable.

Each of these colorings has the rainbow Hamiltonian cycle given by the vertex se-

quence $v_2, v_1, v_6, v_4, v_5, v_3$. □

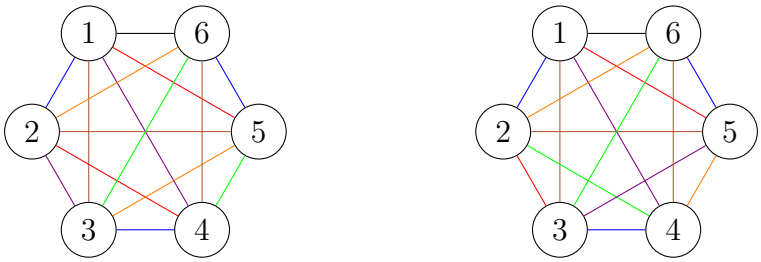


Figure 2.5: Colorings corresponding to the partition 3+3+2+2+2+2+1

Proposition 11. *There are 4 unique 6-colorings of K_6 associated to the partition 15=2+2+2+2+2+2+1. These are the four colorings shown by Figure 2.6.*

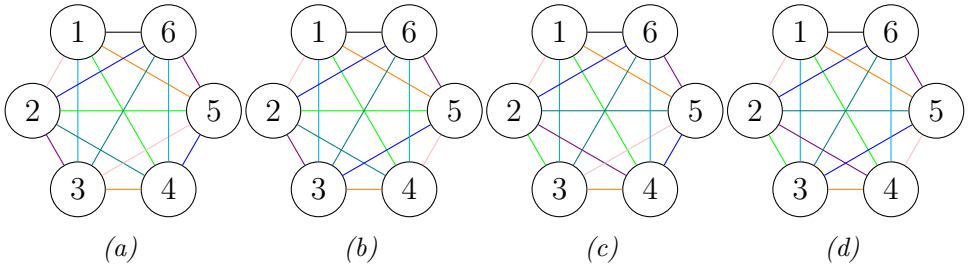


Figure 2.6: Colorings corresponding to the partition 2+2+2+2+2+2+2+1

Proof. First, we note that each of the color classes of size two must contain at least one of the endpoints of the edge colored by the color of size one. So WLOG let the edge colored by the color of size 1 be the edge $\{v_1, v_6\}$. Now since 8 edges are incident to the edge $\{v_1, v_6\}$, and there are seven colors of size 2, WLOG suppose that cyan is the unique color of size 2 which has two edges incident to the edge $\{v_1, v_6\}$. Then let pink orange and green be the other colors with color class of size 2 containing v_1 and the other three colors be those containing edges incident to v_6. WLOG let

orange be the color of the edge $\{v_3, v_4\}$. Now split into cases based on whether the edge $\{v_2, v_5\}$ is colored by a color with color class containing v_1 (like green in (a) and (b)) or a color with color class containing v_6 (WLOG teal in (c) or (d)). In either case $\{v_3, v_5\}$ and $\{v_4, v_5\}$ must be colored by either pink or blue, split into cases again based on whether $\{v_3, v_5\}$ is pink and $\{v_4, v_5\}$ is blue (a), (c) or vice versa (b), (d). The remaining edges are determined uniquely. □

The colorings each have rainbow cycles given by the following vertex sequences $(a) : v_1, v_2, v_3, v_6, v_4, v_5$, $(b) : v_1, v_2, v_5, v_3, v_4, v_6$, $(c) : v_1, v_2, v_3, v_6, v_4, v_5$, and $(d) : v_1, v_2, v_4, v_6, v_3, v_5$.

So we have supported all the results contained in Table 2.1. These results show that there may be many patterns within the structure of unshrinkable colorings which merit further study. It is quite likely that these results can be extended by a computer search and indeed we believe that this direction has merit.

Chapter 3

Edge Ordering: A Turán Problem

3.1 Linear Edge Orderings

We begin by defining a linear edge ordering and other terms we need to study these orderings.

Definition 13. A linear edge ordering on a graph G of size is an injective function $L : E(G) \to \mathbb{R}$.

We say that two edge-orderings L' and L on a graph G are equivalent if $\forall e_1, e_2 \in GL'(e_1) > L'(e_2) \iff L(e_1) > L(e_2)$.

Remark. For any edge ordering L', we can find an equivalent ordering $L : E(G) \to [m]$ which corresponds each edge of G to an integer between 1 and the number of edges of G. We chose this ordering to describe the equivalence class of equivalent orderings.

Following the convention of [18], we write G^L to indicate the graph G with the edge labeling L. For the definition of extremal number, we refine the definition of a subgraph to be a subgraph that respects the linear order. When we have a path,

we write P_4^{132} to represent the path for which reading the labeling of edges from one endpoint to the other yields 1,3,2. Similarly, we use the notation C_4^{1243} to refer to the ordering of C_4 for which reading the labels of the edges around the cycle in some order gives the sequence 1,2,4,3.

Definition 14 (Subgraph)**.** A graph H with an ordering L' is a subgraph of G^L if H is a subgraph of G and $(\forall e_1, e_2 \in E(H))L'(e_1) > L'(e_2) \iff L(e_1) > L(e_2)$.

For extremal number problems, we use the notation $\mathrm{ex}_<(n, H^{L'})$ and $\mathrm{ex}_<(n, \mathcal{H}^{\mathcal{L}'})$ when we want to forbid an linearly edge ordered graph H or collection of linearly edge ordered graphs $\mathcal{H}$ under this new definition of subgraphs. That is to say, $\mathrm{ex}_<(n, H^{L'})$ is the largest possible size of a graph on n vertices which can be equipped with a linear ordering L so that G^L does not contain a copy of $H^{L'}$.

In [18], the authors analyze the extremal numbers $\mathrm{ex}_<(n, H)$ when H is a small path with any of the possible linear orderings. The authors define the max-labeling of K_n, a linear ordering of the complete graph K_n which is good at avoiding linearly ordered subgraphs: Let the vertices be ordered $v_1 < v_2 < ... < v_n$. At each step, for the largest ordered vertex v with an unordered edge, label the edge $\{v, u\}$ with the largest unused label of $[||G||]$, where u is the largest vertex such that $\{v, u\}$ is unlabeled. Repeat this process until the graph is completely labeled. This ordering is shown for K_5 (Fig. 3.1). Towards the goal of finding which orderings are contained in the max-labeling, the authors call a vertex $v \in V(G)$ *close* if every edge labeled between the largest and smallest edge incident to v is also incident to v. For example the vertex v_5 is close in the max-labeling of K_5 (Fig. 3.1). The authors note that one of the endpoints of a max-labeled edge in any graph contained in the max-labeling is

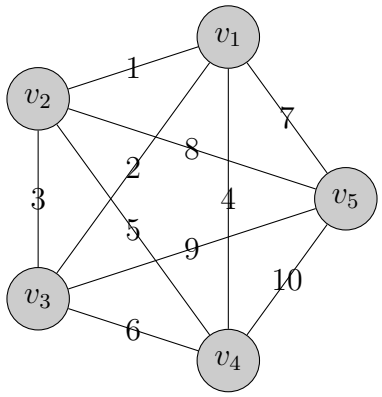


Figure 3.1: Max-labeling of K_5

close.

3.2 Cyclic Edge Orderings

We define another way of ordering the edges of a graph, this time using the elements of some cyclic group. However, instead of trying to rigorously define an edge ordering using a cyclic group, we instead define an equivalent notion by defining a specific family of linear edge orderings. We will define a cyclic shift first.

Definition 15 (Cyclic Shift)**.** Suppose L is some linear ordering on a graph G with size m. Suppose we let L be the ordering with elements from $[m]$ instead of $\mathbb{R}$, (Rmk. 3.1). Then a cyclic shift by c of L generates a new linear permutation L' with $L'(e) = (L(e) + c) \ \% \ m$ where $a \ \% \ m$ is the element between 1 and m in the equivalence class $a \mod m$.

We represent the set of the edge orders of G given by all cyclic shifts of L by the family $\mathcal{C}(G^L)$. That is $\mathcal{C}(G^L)$ represents the set of labelings equivalent to L via cyclic shift each equipped to G.

We note that two families $\mathcal{C}(G^L)$ and $\mathcal{C}(H^{L'})$ are equivalent precisely when some G^l in $\mathcal{C}(G^L)$ is equivalent to some $H^{l'}$ in $\mathcal{C}(H^{L'})$. We want to analyze results for $\mathrm{ex}(n, \mathcal{C}(G^L))$.

3.2.1 Edge Orderings of C_n

We want to be able to find $\mathrm{ex}(n, \mathcal{C}(G^L))$. A good first step is figuring out what $\mathcal{C}(G^L)$ looks like. We will find out how many elements it has. We begin by analyzing a counting problem:

Question 3. *How many unique ways are there to cyclically order the edges of C_n? That is, how many different families $\mathcal{C}(C_n^L)$ are there?*

We know that if two linear orderings are equivalent, they generate the same family $\mathcal{C}$. However, if two linear orderings are equivalent through some cyclic shift, then they will also generate the same $\mathcal{C}$. So we will find the number of unique linear ordering of C_n there are and then find out how many of those are equivalent via some cyclic shift.

Proposition 12. *We have that the number of unique ways to linearly order C_n is $\frac{(n-1)!}{2}$.*

We know that there are $n! = |(S_n)|$ ways to order the elements 1 through n, where S_n is the symmetric group on n elements. But since we are ordering these elements along a rotationally symmetric cycle, we can start read a order starting anywhere (n options) along the cycle. We can also read in either direction (2 options). So there are $n!$ ways divided by n rotations and half of those orders are reflections of each other. Thus we have $(n-1)!/2$ unique linear orderings of C_n. Algebraically, we have the

dihedral group D_{2n} for the symmetries of the cycle so we have $|S_n/D_{2n}|$ unique linear ordering of C_n. We will notate this number as $\mathcal{U}(n)$ for the unique linear orderings of C_n.

Now we will see how many of these linear edge orderings are unique under the cyclic shift operation. That is, how many unique cyclic orderings of C_n are there? For example, in C_6 there are examples when 1,2,3 or 6 elements of the unique linear orderings are equivalent modulo the cyclic shifting actions. Examples of these are the cycles $C_6^{123456}, C_6^{123654}, C_6^{214365}$, and C_6^{126543} respectively. We will label these cycles as class 1, class 2, class 3, and class 6 respectively. For a general $n > 3$, we can find cycles of class d for all divisors d of n. We observe the following relation.

Remark. $\mathcal{U}(n) = \sum_{d \text{s.t.} d|n}(\text{number of cycles of class } d)d$

The classes of size 1 for a general C_n are easy to enumerate, these correspond with cycles for which the label of each edge differs from that of its neighbours by k for all edges in C_n. Such a cycle is possible for all k with k, n coprime, and to avoid double counting let's choose k from the set $[\lfloor n/2 \rfloor]$. The table 3.1 gives the number of unique linear orderings and the number of unique cyclic orderings, as well as thew relationship between these quantities for small cycles.

A surprising proof of part of Wilson's Theorem

For a short aside, we note that there are $p-1/k$ classes of size 1 making up the cyclic colorings of C_p when p is prime. So there are $\frac{(p-1)!}{2} - \frac{p-1}{2} = \frac{(p-1)!-p-1}{2}$ linearly ordered colorings not of class 1 and thus of class p. So there are $\frac{(p-1)!-(p-1)}{2p}$ classes of size p, thus $\frac{(p-1)!-(p-1)}{2p}$ is an integer. This is equivalent to saying $\frac{(p-1)!-(p-1)}{2} \equiv 0 \mod p$.

n	$\mathcal{U}(n)$	Breakdown of $\mathcal{U}(n)$ (classes*class size)	Unique cyclic orderings
3	1	$1*1$	1
4	3	$1*1+1*2$	2
5	12	$2*1+2*5$	4
6	60	$1*1+1*2+5*3+7*6$	14
7	360	$3*1+51*7$	54
8	2520	$2*1+1*2+29*4+300*8$	332
9	$8!/2$	$4*1+5*3+$ the rest $*9$	2246
10	$9!/2$	$2*1+4*2+1150*5+$ the rest $*10$	18264
11	$10!/2$	$5*1+$ the rest $*11$	164950
12	$11!/2$	$2*1+2*2+10*3+39*4$ $+2234*6+$ the rest $*12$	1664354
13	$12!/2$	$6*1+$ the rest $*13$	18423144
prime p	$\frac{(p-1)!}{2}$	$\frac{p-1}{2}*1+\frac{(p-1)!-(p-1)}{2p}*p$	$\frac{(p-1)!-(p-1)/2}{p}+\frac{p-1}{2}$

Table 3.1: Breakdown of distinct linear orderings into classes of cyclic orderings

$\frac{(p-1)!-(p-1)}{2} = ((p-2)!-1)(\frac{p-1}{2})$ so $((p-2)!-1)(\frac{p-1}{2}) \equiv 0 \mod p$. Since $p \not\div \frac{p-1}{2}$ and p is prime, $(p-2)!-1 \equiv 0 \mod p$, thus $(p-2)! \equiv 1 \mod p$, and multiplying by $p-1$ gives us $(p-1)! \equiv p-1 \equiv -1 \mod p$. This is one half of Wilson's theorem.

3.3 Some results on cyclically ordered cycles

We will show that some cyclically-ordered cycles are not contained in the max-labeling of K_n. That is to say, we find some linearly ordered cycles L such that $\mathrm{ex}_>(n, \mathcal{C}(H^L)) = \binom{n}{2}$.

Firstly, consider $C_n^{123\ldots n}$ for some even n. We note that $\{C_n^{123\ldots n}\} = \mathcal{C}(C_n^{123\ldots n})$. This is because a cyclic shift of 1 corresponds to some rotation of the labeling. We also note that if two incident edges $\{u, v\}$, $\{v, w\}$ are labeled by the max-labeling M,

$M(\{u,v\}) > M(\{v,w\}) \iff M_{vert}(u) > M_{vert}(w)$, where M is the edge max-labeling coming from the vertex labeling M_{vert}.

Proposition 13. *When $n > 2$ is even, we claim that* $\text{ex}_{>}(n, \mathcal{C}(C_n^{123\ldots n}) = \binom{n}{2}$

Proof. It suffices to show that $C_n^{123\ldots n}$ is not in the max-labeling of any complete graph. Suppose it is, for $i < n$, let $\{v_i, v_{i+1}\}$ be the edge labeled i in the cycle C_n, where v_i means the i^th vertex along C_n but not any vertex ordering. However, we can say something about the relationship between the labels of the vertices $l(v_k)$ given by the max-labeling. Namely, for $i < k-1$ we have $l(v_i) < l(v_{i+2})$ since this is necessary for the edge $\{v_{i+1}, v_{i+2}\}$ to be labeled larger than $\{v_i, v_{i+1}\}$ under the complete graph labeling. So we have that under the vertex labeling induced by the max labeling $v_1 < v_3 < v_5 < \ldots < v_{n-3} < v_{n-1}$. But we also have $v_1 > v_{n-1}$, since the edge $\{v_1, v_n\}$ is labeled n which is bigger than the labeling of the edge $\{v_n, v_{n-1}\}$, $n-1$. So $v_1 > v_{n-1}$ and $v_1 > v_{n-1}$, a contradiction. So the max-labeled complete graph avoids $C_n^{123\ldots n}$. □

For our next observation, we will extend the notion of close vertex to cyclic orderings.

Definition 16. We say that a vertex is cyclically-close if either every edge labeled between the largest and smallest edge incident to v is also incident to v (the normal sense of close) or if v is incident to both the largest and smallest edge.

Remark. We have that a cyclic shift preserves this new notion of closeness.

Proposition 14. *If C_n^L has no cyclically-close vertices, then* $\text{ex}_{<}(n, \mathcal{C}(C_n^L)) = \binom{n}{2}$.

Proof. Suppose towards a contradiction that $\mathcal{C}(C_n^L)$ is not avoided by the max-labeling. Then fix the linearly ordered subgraph contained in the max labeling $C_n^{L'} \in \mathcal{C}(C_n^L)$, by definition of $\mathcal{C}$, L' is equivalent to L under some cyclic shift. Since C_n^L has no cyclically-close vertices, neither does $C_n^{L'}$. But one of the endpoints v of the edge maximally labeled by L' is close under the normal notion of close by an observation from [18]. Since all close vertices are cyclically-close, our endpoint v is cyclically-close, and we have a contradiction. So the max-labeling avoids all elements of $\mathcal{C}(C_n^L)$ and $\mathrm{ex}_<(n, \mathcal{C}(C_n^L)) = \binom{n}{2}$. □

Bibliography

[1] M. Maamoun and H. Meyniel. On a problem of g. hahn about coloured hamiltonian paths in $K2t$. 51(2):213–214.

[2] Gena Hahn. Un jeu de colouration. In *Actes du Colloque de Cerisy*, volume 12, pages 18–18, 1980.

[3] Geňa Hahn and Carsten Thomassen. Path and cycle sub-ramsey numbers and an edge-colouring conjecture. *Discrete Mathematics*, 62(1):29–33, 1986.

[4] Alan Frieze and Bruce Reed. Polychromatic hamilton cycles. *Discrete Mathematics*, 118(1-3):69–74, 1993.

[5] Michael Albert, Alan Frieze, and Bruce Reed. Multicoloured hamilton cycles. *the electronic journal of combinatorics*, 2(1):R10, 1995.

[6] András Gyárfás, Miklós Ruszinkó, Gábor Sárközy, and Richard H. Schelp. Long rainbow cycles in proper edge-colorings of complete graphs. 50:45–53. Publisher: Centre for Combinatorics.

[7] Saieed Akbari, Omid Etesami, H Mahini, and Mohammad Mahmoody. On rainbow cycles in edge colored complete graphs. *Australasian Journal of Combinatorics*, 37:33, 2007.

[8] Noga Alon, Alexey Pokrovskiy, and Benny Sudakov. Random subgraphs of properly edge-coloured complete graphs and long rainbow cycles. 222(1):317–331.

[9] András Gyárfás and Mehdi Mhalla. Rainbow and orthogonal paths in factorizations of \(K_n\). *Journal of Combinatorial Designs*, 18(3):167–176, 2010.

[10] Heidi Gebauer and Frank Mousset. On Rainbow Cycles and Paths, July 2012. arXiv:1207.0840 [cs, math].

[11] He Chen and Xueliang Li. Long rainbow path in properly edge-colored complete graphs, March 2015. arXiv:1503.04516 [math].

[12] W Mantel. Solution to problem 28, by h. *Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh, and WA Wythoff, Wiskundige Opgaven*, 10:60–61, 1907.

[13] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems.

[14] Paul Turán. On an external problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[15] Paul Erdös and Arthur H Stone. On the structure of linear graphs. 1946.

[16] Peter Brass, Gyula Károlyi, and Pavel Valtr. A turán-type extremal theory of convex geometric graphs. In Boris Aronov, Saugata Basu, János Pach, and Micha Sharir, editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, Algorithms and Combinatorics, pages 275–300. Springer.

[17] Gábor Tardos. EXTREMAL THEORY OF ORDERED GRAPHS. pages 3235–3243. Conference Name: International Congress of Mathematicians 2018 ISBN: 9789813272873 9789813272880 Place: Rio de Janeiro, Brazil Publisher: WORLD SCIENTIFIC.

[18] Dániel Gerbner, Abhishek Methuku, Dániel T. Nagy, Dömötör PÍvölgyi, Gábor Tardos, and Máté Vizer. Turán problems for edge-ordered graphs. 160:66–113.